Lipschitz perturbation of BV sweeping process

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Outline of the talk

Convex sweeping process equation

Sem. Anal. Conv. 1971

• In 1971, J.J. Moreau introduced the "sweeping process" (in the absolutely continuous framework) as the evolution differential inclusion

$$\text{(SP)} \quad \frac{du}{dt}(t) \in -N(C(t); u(t)) \quad \text{for a.e. } t \in [T_0, T], \text{ with } u(T_0) = u_0 \in C(T_0),$$

where $0 \leq T_0 < T < +\infty$; for convenience, we will write sometimes, as usual, $\dot{u}(t)$ in place of $\frac{du}{dt}(t)$.

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CRAS 1971-Sem. Anal. Conv. 1973

• In an earlier paper in 1971 Moreau showed how such an evolution equation arises in the theory of elastic mechanical systems submitted to nonsmooth efforts as *dry friction*; note that the velocity in such cases may present discontinuity in time. He also provided later in a 1973 paper more details on applications to elasticity and other fields of mechanics.

Moreau's references

• J.J. Moreau, Sur l'évolution d'un système élastoplastique, C. R. Acad. Sci. Paris Sér. A-B, 273 (1971), pp. A118-A121.

• J.J. Moreau, Rafle par un convexe variable I, Sém. Anal. Convexe Montpellier (1971), Exposé No 15, 43 pages.

• J.J. Moreau, Rafle par un convexe variable II, Sém. Anal. Convexe Montpellier (1972), Exposé No 3, 32 pages.

• J.J. Moreau, Rétraction d'une multi-application, Sém. Anal. Convexe Montpellier (1972), Exposé No 13, 90 pages.

• J.J. Moreau, On unilateral constraints, friction and plasticity, in "New Variational Techniques in Mathematical Physics", pp. 173-322, C.I.M.E. II Ciclo (1973), Edizioni Cremonese, Roma.

• J.J. Moreau, Sur les mesures différentielles des fonctions vectorielles à variation bornée, Sem. Anal. Convexe Montpellier, (1975), Exposé No 17, 39 pages.

• J.J. Moreau, Evolution problem associated with a moving convex set in a Hilbert space, J. Differential Equations, 26 (1977), pp. 347ñ374.

• J.J. Moreau, Bounded variation in time, Topics in nonsmooth mechanics, (1988), pp. 1-74.

• J.J. Moreau, Numerical aspects of the sweeping process, Comput. Methods Appl. Mech. Engrg., 177 (1999), pp. 329ñ349.

J.J. Moreau: More than 25 papers dedicated by Moreau to the topic "Sweeping Process".

Other contibutions in the absolutely continuous setting

- H. Benabdellah
- F. Bernicot - J. Venel
- B. Brogliato
- C. Castaing
- G. Colombo - V.V. Goncharov
- M. Monteiro-Marques
- M. Bounkhel - L.T.; J.F. Edmond - L.T.; A. Jourani - T. Haddad - L.T.; L.T.
- A. Tolstonogov
- M. Valadier

Moreau theorem on abs. cont. Convex sweeping proc.

The 1971-1973 papers are concerned with the situation where the discontinuity of the velocity is exhibited by an absolute continuity property of the state of the system. The main result of the 1973 paper can be stated as follows.

Theorem (Moreau's theorem for abs. cont. convex sweeping proc.)

Assume that the sets $C(t)$ of the Hilbert space H are nonempty closed convex sets for which there is a nondecreasing absolutely continous function $v(\cdot) : [T_0, T] \to \mathbb{R}_+ := [0, +\infty[$ such that, for each $y \in H$,

$$d(y, C(t)) \leq d(y, C(s)) + v(t) - v(s) \quad \text{for all } T_0 \leq s \leq t \leq T.$$

Then, the evolution equation (SP) admits one and only one absolutely continuous solution.

Moreau theorem on BV convex sweeping proc.

To take into account the more general situation where there are jumps, Moreau tranformed the above model into a measure differential inclusion and proved in 1977 an existence result that we give in the following form.

Theorem (Moreau's theorem for convex sweeping proc. with bounded variation)

Assume that the sets $C(t)$ of the Hilbert space H are nonempty closed convex sets for which there is a positive Radon measure μ on $[T_0, T]$ such that, for each $y \in H$,

$$d(y, C(t)) \leq d(y, C(s)) + \mu(]s, t]), \quad \text{for all } T_0 \leq s \leq t \leq T.$$

Then, the measure differential evolution inclusion

$$\begin{cases} du \in -N(C(t); u(t)) \\ u(T_0) = u_0 \in C(T_0) \end{cases} \tag{1}$$

admits one and only one right continuous solution with bounded variation.

Moreau theorem on BV convex sweeping proc.

Definition

A mapping $u(\cdot) : [T_0, T] \to H$ is a solution of the measure differential inclusion in the theorem provided that it is right continuous with bounded variation with $u(T_0) = u_0$ and $u(t) \in C(t)$ for all $t \in [T_0, T]$ and the differential measure du associated with u admits the derivative measure $\frac{du}{d\mu}$ as a density relative to μ and

$$\frac{du}{d\mu}(t) \in -N(C(t); u(t)) \quad \text{for } \mu - \text{a.e. } t \in [T_0, T].$$

Moreau theorem on BV convex sweeping proc.

Proposition (J.F. Edmond and L.T.)

A mapping $u(\cdot)$ is a solution in the above sense if and only if the latter inclusion is fulfilled with some positive Radon measure ν on $[T_0, T]$ in place of μ.

Other related fields

- Granular material
- Planning procedure
- Electrical circuit
- Crowd motion
- Hysteresis

Hysteresis

Many problems from thermo-plasticity, phase transition (etc) in the literature lead to variational inequalities in the form below. Consider, for example, the following elasto-plastic one (P. Krejči). Let Z be a closed convex set of the $\frac{1}{2}N(N+1)$-dimensional vector space E of symmetric tensors $N \times N$. Assume that the interior of Z is nonempty, so $\mathrm{int}\, Z \neq \emptyset$ corresponds to the elasticity domain and $\mathrm{bdry}\, Z$ to the plasticity. Write the strain tensor $\varepsilon = (\varepsilon)_{i,j}$ (depending on time t) as $\varepsilon := \varepsilon^e + \varepsilon^p$, where ε^e is the elastic strain and ε^p the plastic strain. The elastic strain ε^e is related to the stress tensor $\sigma = (\sigma)_{i,j}$ linearly, that is, $\varepsilon^e = A^2\sigma$, where A is a (constant) symmetric positive definite matrix. The system is then subjected to the variational inequality:

$$\langle \dot{\varepsilon}^p(t), z\rangle \leq \langle \dot{\varepsilon}^p(t), \sigma(t)\rangle,\ \forall z \in Z : \quad \text{principle of maximal dissipation}$$

and to the region constraint $\sigma(t) \in Z$ for all $t \in [0, T]$; in this system, the tensor strain ε is supposed to be given as an absolutely continuous mapping and the initial tensor stress σ_0 is gven in Z.

Hysteresis

Observing that the above inequality can be written as

$$\langle -A\dot{\sigma}(t) + A^{-1}\dot{\varepsilon}(t), A\sigma(t) - Az\rangle \geq 0,$$

Setting

$$\zeta(t) := A\sigma(t) - A^{-1}\varepsilon(t),$$

yields to the equivalent inequality

$$\langle -\dot{\zeta}(t), \zeta(t) - (Az - A^{-1}\varepsilon(t))\rangle \geq 0 \text{ for all } z \in Z.$$

Hysteresis

By setting,

$$C(t) := -A^{-1}\varepsilon(t) + A(Z),$$

the problem is reduced to the convex sweeping process

$$\begin{cases} \dot{\zeta}(t) \in -N(C(t);\zeta(t)) \\ \zeta(0) = A\sigma_0 - A^{-1}\varepsilon(0) \in C(0). \end{cases}$$

Clearly, we have

$$d(x,C(t)) \leq d(x,C(s)) + \|A^{-1}\| \int_s^t \|\dot{\varepsilon}(r)\|\,dr.$$

This provides, according to Moreau Theorem (abs. cont. case), existence and uniqueness of solution for that system.

Hysteresis

This defines a mapping $\Phi : W^{1,1}([0, T], E)$ assigning to each absolutely continuous mapping $\varepsilon \in W^{1,1}([0, T], E)$ the solution $\Phi(\varepsilon) := \sigma_\varepsilon$ of the system. This mapping Φ enjoys two particular properties:

Hysteresis

• Rate independence:

Denoting by σ_ε the solution associated with ε and taking any absolutely continuous increasing bijection $\theta : [0,T] \to [0,T]$, for almost every $t \in [0,T]$, we have $\langle -A^2\dot{\sigma}_\varepsilon(\theta(t)) + \dot{\varepsilon}(\theta(t)), \sigma_\varepsilon(\theta(t)) - z\rangle \geq 0$ hence

$$\langle -A^2\dot{\sigma}_\varepsilon(\theta(t))\dot{\theta}(t) + \dot{\varepsilon}(\theta(t))\dot{\theta}(t)), \sigma_\varepsilon(\theta(t)) - z\rangle \geq 0$$

(since $\dot{\theta}(t) \geq 0$ a.e.).

Hysteresis

From the latter, one obtains, for almost every $t \in [0, T]$,

$$\begin{cases} \langle -A^2 \frac{d}{dt}(\sigma_\varepsilon \circ \theta)(t) + \frac{d}{dt}(\varepsilon \circ \theta)(t), (\sigma_\varepsilon \circ \theta)(t) - z \rangle \geq 0 \\ (\sigma_\varepsilon \circ \theta)(0) = \sigma_0 \in C(0). \end{cases}$$

The uniqueness property guarantees that $\sigma_\varepsilon \circ \theta$ is the solution associated with $\varepsilon \circ \theta$, otherwise stated, $\Phi(\varepsilon \circ \theta) = \Phi(\varepsilon) \circ \theta$. The latter equality is known in the literature as the rate independence property.

Hysteresis

• Causality

For each $\tau \in [0,T]$ and $\varepsilon \in W^{1,1}([0,T],E)$, denoting by σ_ε the solution on $[0,T]$ of the system above, the restriction of $\sigma|_{[0,\tau]}$ to $[0,\tau]$ coincides with the solution on $[0,\tau]$ of the system associated with $\varepsilon|_{[0,\tau]}$ according to the same uniqueness property above. Then, for $\varepsilon_i \in W^{1,1}([0,T],E)$ ($i = 1,2$), we have

$$\varepsilon_1|_{[0,\tau]} = \varepsilon_2|_{[0,\tau]} \;\Rightarrow\; \sigma_{\varepsilon_1}(t) = \sigma_{\varepsilon_2}(t)\;\forall t \in [0,\tau],$$

otherwise stated

$$\varepsilon_1|_{[0,\tau]} = \varepsilon_2|_{[0,\tau]} \;\Rightarrow\; \Phi(\varepsilon_1)(t) = \Phi(\varepsilon_2)(t)\;\forall t \in [0,\tau];$$

this kind of property is generally called the causality property in the literature.

Hysteresis

Hysteresis operator.

• Both rate independence and causality properties translate that Φ is an hysteresis operator according to the following references where those properties are brought to light with various physical examples with hysteresis phenomena:

• M. Brokate and J. Sprekels, Hysteresis and Phase Transitions, Applied Mathematical Sciences, Vol 121, Springer-Verlag, Berlin, 1996.
• P. Krejčı, Vector hysteresis models, European J. Appl. Math. 2 (1991), 281-292.
• A. Visintin, Differential Models of Hysteresis, Applied Mathematical Sciences, Vol 111, Spinger-Verlag, Berlin, 1994.

For several other models, we refer to Brokate-Sprekels (above). Of course, by Theorem 1.1 the mathematical features and properties above still hold in the context of a Hilbert space H with any closed convex set Z (without any condition on its interior) and any coercive bijective bounded symmetric linear operator $A : H \to H$.

Outline of the talk

Lipschitz perturbation

We come now to the *differential inclusion*

$$\text{(LPSP)} \quad \begin{cases} du \in -N(C(t); u(t)) + f(t, u(t)), \\ u(0) = u_0 \in C(0) \end{cases} \tag{2}$$

where $f : I \times H \to H$ is a Carathéodory mapping and where the variation of $C(t)$ is translated by a given positive Radon measure μ on I as in the second aforementioned Moreau Theorem.

Lipschitz perturbation

The case of a set-valued mapping $F : I \times H \rightrightarrows H$ (in place of f) has been studied by C.Castaing and M. Monteiro-Marques in the finite dimensional setting and by J.F. Edmond and L.T. under the assumption $F(t,x) \subset \beta(t)(1+\|x\|)K$ where K is a fixed normed compact subset of H. Our aim here is to study in the Hilbert setting the new variant where f satisfies a Lipschitz condition and no compactness condition is assumed.

Lipschitz perturbation

Abs. cont. measures

- For two positive Radon measures ν and $\hat{\nu}$ on I and for $I(t,r) := I \cap [t-r, t+r]$, it is known that the limit

$$\frac{d\hat{\nu}}{d\nu}(t) := \lim_{r \downarrow 0} \frac{\hat{\nu}(I(t,r))}{\nu(I(t,r))}$$

(with the convention $\frac{0}{0} = 0$) exists and is finite for ν-almost every $t \in I$ and it defines a Borel function of t, called the derivative of $\hat{\nu}$ with respect to ν.

Lipschitz perturbation

Abs. cont. measures

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Density.

- Furthermore, the measure $\hat{\nu}$ is absolutely continuous with respect to ν if and only if $\frac{d\hat{\nu}}{d\nu}(\cdot)$ is a density of $\hat{\nu}$ relative to ν, or otherwise stated, if and only if the equality $\hat{\nu} = \frac{d\hat{\nu}}{d\nu}(\cdot)\nu$ holds true.

Lipschitz perturbation

Integ/abs.cont.

• Under such an absolute continuity assumption, a mapping $u(\cdot) : I \to H$ is $\hat{\nu}$-integrable on I if and only if the mapping $t \mapsto u(t)\frac{d\hat{\nu}}{d\hat{\nu}}(t)$ is ν-integrable on I; furthermore, in that case,

$$\int_I u(t)\,d\hat{\nu}(t) = \int_I u(t)\frac{d\hat{\nu}}{d\nu}(t)\,d\nu(t).$$

Lipschitz perturbation

Integ/abs.cont.

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Abs. cont. equiv.

• When ν and $\hat{\nu}$ are each one absolutely continuous with respect to the other, we will say that they are absolutely continuously equivalent.

Lipschitz perturbation

Suppose that the mapping $u(\cdot) : I \to H$ has *bounded variation* and denote by du the differential measure associated with u; if in addition, $u(\cdot)$ is right continuous, then

$$u(t) = u(s) + \int_{]s,t]} du \quad \text{for all } s, t \in I \text{ with } s \leq t.$$

Conversely, if there exists some mapping $\hat{u}(\cdot) \in L^1_\nu(I, H)$ such that $u(t) = u(T_0) + \int_{]T_0,t]} \hat{u}\, d\nu$ for all $t \in I$, then $u(\cdot)$ is of bounded variation and right continuous and $du = \hat{u}\, d\nu$; so $\hat{u}(\cdot)$ is a density of the vector measure du relative to ν. Then, putting $I^-(t,r) := [t-r, t]$ and $I^+(t,r) := [t, t+r]$, according to a result of Moreau and Valadier for ν-almost every $t \in I$, the following limits exist in H and

$$\hat{u}(t) = \frac{du}{d\nu}(t) := \lim_{r \downarrow 0} \frac{du(I(t,r))}{d\nu(I(t,r))} = \lim_{r \downarrow 0} \frac{du(I^-(t,r))}{d\nu(I^-(t,r))} = \lim_{r \downarrow 0} \frac{du(I^+(t,r))}{d\nu(I^+(t,r))}. \tag{3}$$

In particular, the last equality ensures that

$$\frac{du}{d\nu}(t) = \frac{du(\{t\})}{d\nu(\{t\})} \quad \text{and} \quad \frac{d\lambda}{d\nu}(t) = 0, \text{ whenever } \nu(\{t\}) > 0. \tag{4}$$

Above and in the rest of the paper λ denotes the Lebesgue measure.

Lipschitz perturbation

Definition

A mapping $u : I \to H$ is a solution of the measure differential inclusion (LPSP) if:
(i) $u(\cdot)$ is of bounded variation, right continuous, and satisfies $u(T_0) = u_0$ and $u(t) \in C(t)$ for all $t \in I$;
(ii) there exists a positive Radon measure ν absolutely continuously equivalent to $\mu + \lambda$ and with respect to which the differential measure du of $u(\cdot)$ is absolutely continuous with $\frac{du}{d\nu}(\cdot)$ as an $L^1_\nu(I, H)$-density and

$$\frac{du}{d\nu}(t) + f(t, u(t))\frac{d\lambda}{d\nu}(t) \in -N(C(t); u(t)) \quad \nu - a.e.\ t \in I.$$

Lipschitz perturbation

The following proposition concerning a particular chain rule for differential measures will be needed. Its statement is a consequence of a more general result from J.J. Moreau.

Proposition

Let H be a Hilbert space, ν be a positive Radon measure on the closed bounded interval I, and $u(\cdot) : I \to H$ be a right continuous with bounded variation mapping such that the differential measure du has a density $\frac{du}{d\nu}$ relative to ν. Then, the function $\Phi : I \to \mathbb{R}$ with $\Phi(t) := \|u(t)\|^2$ is a right continuous with bounded variation function whose differential measure $d\Phi$ satisfies, in the sense of ordering of real measures,

$$d\Phi \leq 2\langle u(\cdot), \frac{du}{d\nu}(\cdot)\rangle d\nu.$$

LIpschitz perturbation

The next result is a substitute of Grownwall's lemma relative to Radon measures.

Lemma

Let ν be a positive Radon measure on $[T_0, T]$ and let $g(\cdot) \in L^1_\nu([T_0, T], \mathbb{R}_+)$. Assume that, for a fixed real number $\theta \geq 0$, one has, for all $t \in]T_0, T]$,

$$0 \leq g(t)\nu(\{t\}) \leq \theta < 1.$$

Let $\varphi \in L^\infty_\nu([T_0, T], \mathbb{R}_+)$ and let some fixed real number $\alpha \geq 0$ satisfying, for all $t \in [T_0, T]$,

$$\varphi(t) \leq \alpha + \int_{]T_0, T]} g(s)\varphi(s)\, d\nu(s).$$

Then, for all $t \in [T_0, T]$,

$$\varphi(t) \leq \alpha \exp\{\frac{1}{1-\theta} \int_{]T_0, t]} g(s)\varphi(s)\, d\nu(s)\}.$$

Lipschitz perturbation

A stability property of the subdifferential of the distance function from a continuous moving set.

Proposition (S. Adly, T. Haddad, L.T.)

Let E be a metric space, $C : E \rightrightarrows H$ be a set-valued mapping with nonempty closed convex sets of a normed space X, and let $t_0 \in \mathrm{cl}\, Q$ with $Q \subset E$. Assume that there exists a non-negative real-valued function $\eta : Q \to \mathbb{R}$ with $\lim_{Q \ni t \to t_0} \eta(t) = 0$ such that, for all $t \in Q$,

$$d(x, C(t)) \leq d(x, C(t_0)) + \eta(t) \quad \text{for all } x \in X.$$

Let $(t_n)_n$ be a sequence in Q tending to t_0 and let $(x_n)_n$ be a sequence in H converging to some $x \in C(t_0)$ with $x_n \in C(t_n)$ for all n. Then, for all $z \in X$,

$$\limsup_{n \to \infty} d'_{C(t_n)}(x_n; z) \leq d'_{C(t_0)}(x; z).$$

Above $d'_{C(t_0)}(x; \cdot)$ denotes the usual directional derivative of $d_{C(t_0)}$.

Lipschitz perturbation

Theorem (S. Adly, T. Haddad, L.T.)

Let H be a Hilbert and $C(\cdot) : [T_0, T] \rightrightarrows H$ be a set-valued map from $I := [T_0, T]$ into closed convex sets of H for which there is a positive Radon measure μ on I such that

$$d(y, C(t)) \leq d(y, C(s)) + \mu(]s, t]) \quad \forall\, y \in H,\ \forall s, t \in I \text{ with } s \leq t.$$

Let $f : I \times H \to H$ be a mapping such that

(i) *there exists a non-negative function $\beta(\cdot) \in L^1_\lambda(I, \mathbb{R})$ such that*

$$\|f(t,x)\| \leq \beta(t)(1 + \|x\|) \quad \text{for all } x \in \bigcup_{t \in I} C(t);$$

(ii) *the function $f(\cdot, x)$ is measurable and, for each real $r > 0$, there exists some non-negative function $L_r(\cdot) \in L^1_\lambda(I, \mathbb{R})$ such that*

$$\|f(t,x_1) - f(t,x_2)\| \leq L_r(t)\|x_1 - x_2\| \quad \text{for all } t \in I,\ x_i \in r\mathbb{B}_H.$$

Then, for each $u_0 \in C(T_0)$, the following perturbed sweeping process

$$(\text{LPSP}) \begin{cases} -du \in N(C(t); u(t)) + f(t, u(t)) \\ u(T_0) = u_0 \end{cases}$$

has one and only one right continuous with bounded variation solution.

Lipschitz perturbation

Consider $f = 0$ for a moment and $T_0 = 0 < t_1 < T$ and

$$C(t) = C_1 \quad \text{if } t \in [0, t_1[\quad \text{and} \quad C(t) = C_2 \quad \text{if } t \in [t_1, T].$$

On $[0, t_1[$

• One can check that the variation function of $C(\cdot)$ is given by $\mathrm{var}_C(\cdot) = \mathrm{Haus}(C_0, C_1) \times \mathbf{1}_{[t_1, T]}(\cdot)$. It is a right continuous function with bounded variation. The associated Radon measure is $\mu := \mathrm{Haus}(C_0, C_1)\delta_{t_1}$, where δ_{t_1} is the Dirac mass at t_1. Let us look for solution $u(\cdot)$ on I of

$$-du \in N\big(C(t); u(t)\big) \quad \text{with} \quad u(0) = x_0 \in C_0.$$

Obviously, we must have $u(t) = x_0$ for all $t \in [0, t_1[$.

Lipschitz perturbation

On $[t_1, T]$.

- What about $u(t_1)$? Since $\mu(\{t_1\}) \neq 0$, me must have, for the solution $u(\cdot)$,

$$\frac{du}{d\mu}(t_1) \in -N\big(C(t_1); u(t_1)\big).$$

Since $u(t_1^-) = x_0$, we obtain

$$\frac{u(t_1) - x_0}{\mu(\{t_1\})} = \frac{du}{d\mu}(t_1) \in -N\big(C(t_1); u(t_1)\big) \quad \Longleftrightarrow \quad x_0 - u(t_1) \in N\big(C_1; u(t_1)\big).$$

Lipschitz perturbation

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On $[t_1, T]$.

• Thus, the choice for $u(t_1)$ must be $x_1 := \mathrm{proj}_{C_1}(x_0)$; and one verifies that, putting $u(t) := x_0$ if $t \in [0, t_1[$ and $u(t) := x_1$ if $t \in [t_1, T]$, we get the BV solution.

Lipschitz perturbation

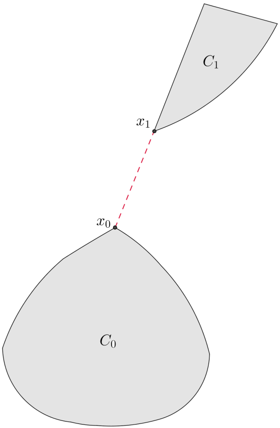

Figure: BV sweeping

Outline of the talk

Construction

Step I. Let us suppose that

$$\int_{T_0}^{T} (\beta(s)+1)\, d\lambda(s) \leq 1/4, \tag{5}$$

and let us construct a sequence of appropriate right continuous with bounded variation mappings.
Put

$$\ell := 2\big(\mu(]T_0,T]) + \|u_0\| + 1\big),$$

and consider on I the positive Radon measure

$$\nu := \mu + (\ell+1)(\beta(\cdot)+1)\lambda. \tag{6}$$

Consider the function $\mathscr{V}(\cdot) : I \to \mathbb{R}$ defined by

$$\mathscr{V}(t) := \nu(]T_0,t])$$

and set

$$V := \mathscr{V}(T) = \nu(]T_0,T]).$$

The function $\mathscr{V}(\cdot)$ is increasing and right continuous with $\mathscr{V}(T_0) = 0$.

Construction

Let $(\varepsilon_n)_n$ be a sequence of positive reals with $\varepsilon_n \downarrow 0$. Let, for each $n \in \mathbb{N}$, a partition $0 = V_0^n < V_1^n < \cdots < V_{q_n}^n = V$ such that

$$V_{j+1} - V_j \leq \varepsilon_n \, \forall j = 0, \cdots, q_n - 1, \quad \text{and} \quad \{V_0^n, \cdots, V_{q_n}^n\} \subset \{V_0^{n+1}, \cdots, V_{q_{n+1}}^{n+1}\}. \tag{7}$$

Put $V_{1+q_n}^n := V + \varepsilon_n$. For each $n \in \mathbb{N}$, consider the partition of I associated with the subsets

$$J_j^n := \mathscr{V}^{-1}([V_j^n, V_{j+1}^n[), \; j = 0, 1, \cdots, q_n,$$

and note that $(J_j^m)_{j=0}^{q_m}$ is a refinement of $(J_j^n)_{j=0}^{q_n}$ whenever $m \geq n$. Since $\mathscr{V}(\cdot)$ is increasing and right continuous, it is easy to see that, for each $j = 0, 1, \cdots, q_n$, the set J_j^n is either empty or an interval of the form $[r, s[$ with $r < s$. Furthermore, we have $J_{q_n}^n = \{T\}$.

Construction

This produces an integer $p(n) \in \mathbb{N}$ and a finite sequence

$$T_0 = t_0^n < t_1^n < \cdots < t_{p(n)}^n = T$$

such that, for each $i \in \{0, \cdots, p(n)-1\}$, there is some $j \in \{0, \cdots, q_n - 1\}$ for which $[t_i^n, t_{i+1}^n[= J_j^n$. It ensues that, for any $i \in \{0, \cdots, p(n)-1\}$,

$$v(]t_i^n, t]) = \mathscr{V}(t) - \mathscr{V}(t_i^n) \leq \varepsilon_n \quad \text{for all } t \in [t_i^n, t_{i+1}^n[, \tag{8}$$

which entails

$$v(]t_i^n, t_{i+1}^n[) \leq \varepsilon_n, \tag{9}$$

hence (since $\lambda \leq v$)

$$t_{i+1}^n - t_i^n \leq \varepsilon_n. \tag{10}$$

For each $i \in \{0, \cdots, p(n)-1\}$, put

$$\sigma_i^n := (l+1) \int_{t_i^n}^{t_{i+1}^n} (\beta(s)+1)\, d\lambda(s) \quad \text{and} \quad \eta_i^n := t_{i+1}^n - t_i^n, \tag{11}$$

and observe that $\eta_i^n \to 0$ as $n \to \infty$.

Construction

For each $i \in \{0, \cdots, p(n)-1\}$, choose some $s_i^n \in [t_i^n, t_{i+1}^n[$ such that

$$\beta(s_i^n) \leq \inf_{s \in [t_i^n, t_{i+1}^n[} \beta(s) + 1, \tag{12}$$

and define the function $\rho_n : I \to I$ by $\rho_n(T) := s_{p(n)-1}^n$ and

$$\rho_n(t) := s_i^n \text{ if } t \in [t_i^n, t_{i+1}^n[\ \ (0 \leq i \leq p(n)-1). \tag{13}$$

Now, put $u_0^n := u_0$, $y_0^n := f(\rho_n(t_0^n), u_0^n)$ and $u_1^n := \mathrm{proj}_{C(t_1^n)}(u_0^n - \eta_0^n y_0^n)$, and define by induction $\{u_i^n : i = 0, \cdots, p(n)\}$ and $\{y_i^n : i = 0, \cdots, p(n)-1\}$ such that

$$y_i^n := f(\rho_n(t_i^n), u_i^n) \quad \text{and} \quad u_{i+1}^n := \mathrm{proj}_{C(t_{i+1}^n)}(u_i^n - \eta_i^n y_i^n). \tag{14}$$

Construction

Define the mapping $u_n(\cdot) : I \to H$ by $u_n(T) := u^n_{p(n)}$ and

$$u_n(t) = u^n_i + \frac{\nu(]t^n_i, t])}{\nu(]t^n_i, t^n_{i+1}])}(u^n_{i+1} - u^n_i + \eta^n_i y^n_i) - (t - t^n_i)y^n_i \quad \text{if } t \in [t^n_i, t^n_{i+1}[. \tag{15}$$

We observe that $u_n(\cdot)$ is well defined on I and it is right continuous with bounded variation on each interval $[t^n_i, t^n_{i+1}]$, so it is right continous with bounded variation on the whole interval I. Furthermore, the definition of $u_n(\cdot)$ can be rewritten, for any $t \in I$, as

$$u_n(t) = u_n(T_0) + \int_{]T_0,t]} \Pi_n(s)\, d\nu(s) - \int_{]T_0,t]} f(\rho_n(s), u_n(\delta_n(s))\, d\lambda(s),$$

where

$$\Pi_n(t) := \sum_{i=0}^{p(n)-1} \frac{u^n_{i+1} - u^n_i + \eta^n_i y^n_i}{\nu(]t^n_i, t^n_{i+1}])} \mathbf{1}_{]t^n_i, t^n_{i+1}]}(t),$$

and $\delta_n(s) := t^n_i$ if $t \in [t^n_i, t^n_{i+1}[$ and $\delta_n(T) := t^n_{p(n)-1}$.

Construction

Since the measure λ is absolutely continuous with respect to ν, it has $\frac{d\lambda}{d\nu}(\cdot)$ as a density in $L^\infty_\nu(I,\mathbb{R}_+)$ relative to ν and then by what precedes, for every $t \in I$,

$$u_n(t) = u_n(T_0) + \int_{]T_0,t]} \{\Pi_n(s) - f(\rho_n(s), u_n(\delta_n(s)))\frac{d\lambda}{d\nu}(s)\}\, d\nu(s).$$

This tells us that the vector measure du_n has the latter integrand as a density in $L^\infty_\nu(I,H)$ relative to ν, so

$$\frac{du_n}{d\nu}(\cdot) \text{ is a density of } du_n \text{ with respect to } \nu, \tag{16}$$

and, for ν-almost every $t \in I$,

$$\frac{du_n}{d\nu}(t) + f(\rho_n(t), u_n(\delta_n(t)))\frac{d\lambda}{d\nu}(t) = \Pi_n(t) = \sum_{i=0}^{p(n)-1} \frac{u^n_{i+1} - u^n_i + \eta^n_i y^n_i}{\nu(]t^n_i, t^n_{i+1}])} \mathbf{1}_{]t^n_i, t^n_{i+1}]}(t). \tag{17}$$

It results that

$$\|\frac{du_n}{d\lambda}(t) + f(\rho_n(t), u_n(\delta_n(t)))\frac{d\lambda}{d\nu}(t)\| \le 1 \quad \nu - \text{a.e. } t \in I. \tag{18}$$

Lipschitz perturbation

Theorem (S. Adly, T. Haddad, L.T.)

Let H be a Hilbert and $C(\cdot) : [T_0,\infty) \rightrightarrows H$ be a set-valued map from $I := [T_0,\infty)$ into closed convex sets of H for which there is a positive Radon measure μ on I such that

$$d(y,C(t)) \leq d(y,C(s)) + \mu(]s,t]) \quad \forall\, y \in H,\ \forall s,t \in I \text{ with } s \leq t.$$

Let $f : I \times H \to H$ be a mapping such that

(i) *there exists a non-negative function $\beta(\cdot) \in L^1_{\lambda,\mathrm{loc}}(I,\mathbb{R})$ such that*

$$\|f(t,x)\| \leq \beta(t)(1+\|x\|) \quad \text{for all } x \in \bigcup_{t \in I} C(t);$$

(ii) *the function $f(\cdot,x)$ is measurable and, for each real $r > 0$, there exists some non-negative function $L_r(\cdot) \in L^1_{\lambda,\mathrm{loc}}(I,\mathbb{R})$ such that*

$$\|f(t,x_1) - f(t,x_2)\| \leq L_r(t)\|x_1 - x_2\| \quad \text{for all } t \in I,\ x_i \in r\mathbb{B}_H.$$

Then, for each $u_0 \in C(T_0)$, the following perturbed sweeping process

$$(\mathrm{LPSP}) \begin{cases} -du \in N(C(t); u(t)) + f(t,u(t)) \\ u(T_0) = u_0 \end{cases}$$

has one and only one right continuous with locally bounded variation solution.