Perturbed sweeping process with a subsmooth set depending on the state

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March 4, 2016

Abstract

The class of subsmooth sets strictly contains the class of closed convex sets and the class of prox-regular sets. The present paper is concerned with the study of perturbed sweeping process differential inclusions where the moving set is nonconvex and non prox-regular and depends both on the time and on the state. We prove the existence of solution, in particular, under the subsmoothness of the moving set.

Keyword : Differential inclusion; Sweeping process; Normal cone; Prox-regular set; Subsmooth set; Subdifferential

2010 Mathematics Subject Classification: primary 34A60, 49J52; secondary 49J24, 49J53

Introduction

In this paper, given a Hilbert space H, we discuss the existence of solution of the evolution process differential inclusion of the form

$$(\mathcal{D}) \qquad \begin{cases} \dot{u}(t) \in -N_{C\big(t,u(t)\big)}\big(u(t)\big) + G\big(t,u(t)\big) & \text{a.e. } t \in [0,T], \\ u(t) \in C\big(t,u(t)\big) \quad \forall t \in [0,T], \\ u(0) = u_0 \in C(0,u_0). \end{cases}$$

In $(\mathcal{D})$, $C : [0,T] \times H \rightrightarrows H$ is a multimapping with nonempty closed values and $G : [0,T] \times H \rightrightarrows H$ is a multimapping with nonempty closed convex values; by $N_{C\big(t,u(t)\big)}(\cdot)$ we denote a normal cone to the set $C(t,u(t))$. As stated, the set $C(t,x)$ depends both on the time t and on the state x. Such differential inclusions have been introduced, for a time-dependent set, in the form

$$(\mathcal{SP}) \qquad \begin{cases} \dot{u}(t) \in -N_{K(t)}\big(u(t)\big) & \text{a.e. } t \in [0,T], \\ u(t) \in K(t) \quad \forall t \in [0,T], \\ u(0) = u_0 \in K(0), \end{cases}$$

by J. J. Moreau [18, 19, 20] who called $(\mathcal{SP})$ a sweeping process because of the mechanical interpretation (see, e.g., [18, 19, 20]). When G is a single-valued mapping and $C(t,x)$ is convex, the definition of normal cone of convex sets makes clear that $(\mathcal{D})$ is a reformulation of quasi-variational inequalities. This leads to also see, when $C(t,x)$ is nonconvex, $(\mathcal{D})$ as an extended quasi-variational inequality.

The study of inclusion $(\mathcal{D})$ probably began with K. Chraibi's thesis [9] with convex sets $C(t,x)$ in the particular space $\mathbb{R}^3$. The second work has been realized in the Hilbert setting by M. Kunze and M. D. P. Monteiro Marques [16] with $G \equiv \{0\}$ and $C(t,x)$ convex for all $t \in [0,T]$ and all $x \in H$. In [8], G is a Carathéodory (single-valued) mapping, that is, measurable with respect to the first variable and continuous with respect to the second one. Associating with each absolutely continuous mapping $y : [0,T] \to H$, with $y(0) = u_0$, the unique solution $\phi(y)$ of the time-dependent sweeping process (with unknown mapping u)

$$\dot{u}(t) \in -N_{C\big(t,y(t)\big)}\big(u(t)\big) + G\big(t,y(t)\big) \quad \text{with } u(0) = u_0 \in C\big(0,y(0)\big),$$

N. Chemetov and M. D. P. Monteiro Marques [8, Theorem 2], by applying the classical Schauder fixed point theorem to the restriction of ϕ over a subtle suitable compact convex set of mappings, proved the existence of solution of $(\mathcal{D})$, for nonconvex prox-regular and ball-compact sets $C(t,x)$ moving in a contractive way with respect to the state x. To be more precise, in [8], it is assumed that there exists an absolutely continuous function $\vartheta : [0,T] \to H$, which is monotone increasing, and a constant $L \in [0,1[$, such that

$$|d\big(y, C(t,x)\big) - d\big(y', C(s,x')\big)| \le \|y - y'\| + \vartheta(t) - \vartheta(s) + L\|x - x'\| \qquad (0.1)$$

for all $t, s \in [0,T]$ with $s < t$ and $x, x', y, y' \in H$. In [4, Theorem 3.3] C. Castaing, A. G. Ibrahim and M. Yarou obtained, under (0.1) and under the prox-regularity and ball-compactness assumption for $C(t,x)$, the existence of solution for $(\mathcal{D})$ when $G \equiv \{0\}$ via another method applying a generalized version of the Schauder fixed point theorem from [15, 23]. Given $t_i^n = iT/n$ (with $i = 0 \cdots, n$) and $\varepsilon_i^n := \vartheta(t_{i+1}^n) - \vartheta(t_i^n)$, the authors in [4] considered the implicit scheme

$$u_0^n = u_0, \quad u_{i+1}^n = \mathrm{proj}_{\,C(t_{i+1}^n, u_{i+1}^n)}(u_i^n), \quad u_{i+1}^n \in B[u_i^n, \varepsilon_i^n/(1-L)].$$

The deep and nice arguments in [4], justifying the existence of such a point u_{i+1}^n via a fixed point theorem from [15, 23], used for each fixed element z in an appropriate subset of H, the continuity of the mapping $x \mapsto \mathrm{proj}_{\,C(t,x)}(z)$ due to (0.1) and to the prox-regularity of $C(t,x)$. Furthermore, with $G \not\equiv \{0\}$ and $C(t,x)$ convex and ball-compact, using a careful adaptation of their method, the authors also showed in the same paper [4] an existence result for $(\mathcal{D})$ with delay, that is, G is an upper semicontinuous and bounded multimapping defined on $[0,T] \times \mathcal{C}_H(-r,0)$ and taking on weakly compact convex values of H; by $\mathcal{C}_H(-r,0)$ we denote with $r > 0$ the space of all continuous mappings from $[-r,0]$ to H; this provides in [4, Corollary 3.1] a solution for $(\mathcal{D})$ under (0.1) and under the convexity of $C(t,x)$. Note that second order sweeping processes with prox-regular sets are also studied in [4]. We refer to D. Azzam-Laouir, S. Izza and L. Thibault [2] for a reduction approach of $(\mathcal{D})$ to an unconstrained differential inclusion when $C(t,x)$ is prox-regular, G is a multimapping, and H is finite dimensional. J. Noel and L. Thibault [22] proved the existence of a solution for $(\mathcal{D})$ in the Hilbert setting when $C(t,x)$ is a ball-compact prox-regular set and G is a multimapping; the method in [22] is an adaptation of the above implicit scheme of [4] via a result on the Hölder property of the metric projection to prox-regular set with respect to the Hausdorff-Pompeiu distance. With the sets $C(t,x)$ prox-regular and contained in a fixed compact set and through the semi-explicit scheme

$$u_0^n = u_0, \quad u_{i+1}^n = \mathrm{proj}_{\,C(t_{i+1}^n, u_i^n)}(u_i^n - \frac{T}{2^n} g_i^n)$$
$$\text{with } g_i^n \in G(t_i^n, u_i^n), \text{ where } t_i^n := i\frac{T}{2^n},\ i = 0, \cdots, 2^n - 1,$$

T. Haddad [14] gave another approach which yields to a proof of existence in the Hilbert setting for $(\mathcal{D})$ without the use of any fixed point theorem. The latter scheme has been also previously used in K. Chraibi [9] in a less large context and under the convexity of $C(t,x)$.

In the present paper [1], using ideas from [14] and [22] we provide a constructive proof of existence of solution for $(\mathcal{D})$ when the sets $C(t,x)$ are ball-compact and subsmooth. The method also allows us to relax, for the multimapping G, the growth conditions which are assumed in [4, 14, 8]. The class of subsmooth sets introduced in [1] strictly contains the class of closed convex sets and the class of prox-regular sets. In the first section, we recall some variational concepts and some properties of subsmooth sets, and we prove an upper semicontinuity result which will be used in our development. The second section is devoted to the aforementioned constructive proof (using no fixed-point theorem) of the differential inclusion $(\mathcal{D})$ governed by subsmooth sets $C(t,x)$.

1 Preliminaries

Throughout the paper, H is a Hilbert space whose inner product is denoted by $\langle\cdot,\cdot\rangle$ and the associated norm by $\|\cdot\|$. The closed unit ball of H with center 0 will be denoted by $\mathbb{B}$, and $B(x,\eta)$ (respectively, $B[x,\eta]$) is the open (respectively, closed) ball of center $x\in H$ and radius $\eta>0$. Given a real $T>0$, by $\mathcal{C}_H(0,T)$ we shall mean the space of all continuous mappings from $[0,T]$ to H. Let S be a nonempty subset of H. For an element $x\in H$, the real $d(x,S)$ or $d_S(x):=\inf\{\|y-x\|:y\in S\}$ is the distance of x from the set S. The projection set of x into S (or the set of nearest points of S to x) is the set

$$\mathrm{Proj}_S(x):=\{y\in S: d_S(x)=\|x-y\|\}.$$

This set is nonempty when S is nonempty and ball-compact; if $\mathrm{Proj}_S(x)$ is a singleton, its unique point will be denoted (as usual) by $\mathrm{proj}_S(x)$. Recall that the subset S of $(H,\|\cdot\|)$ is *ball-compact* provided that $S\cap r\mathbb{B}$ is compact in $(H,\|\cdot\|)$ for every real $r>0$. Obviously any ball-compact set is norm closed, and in finite dimensions S is ball-compact if and only if it is closed. When $y\in\mathrm{Proj}_S(x)$, then we have $x-y\in N^p_S(y)$ where $N^p_S(\cdot)$ denotes the proximal normal cone of S (see (1.4) below and [11] for details).

For a nonempty interval $\mathcal{J}$ of $\mathbb{R}$, we recall that a multimapping $F:\mathcal{J}\rightrightarrows H$ is called Lebesgue measurable if for each open set $U\subset H$ the set $F^{-1}(U):=\{t\in\mathcal{J}: F(t)\cap U\neq\emptyset\}$ is Lebesgue measurable. When the values of F are closed subsets of H, we know (see [7]) that the Lebesgue measurability of F is equivalent to the measurability of the graph of F, that is,

$$\mathrm{gph}\, F\in\mathcal{L}(\mathcal{J})\otimes\mathcal{B}(H),$$

where $\mathcal{L}(\mathcal{J})$ denotes the Lebesgue σ-field of $\mathcal{J}$, $\mathcal{B}(H)$ the Borel σ-field of H, and

$$\mathrm{gph}\, F:=\big\{(t,u)\in\mathcal{J}\times H: u\in F(t)\big\}.$$

[1] The paper is mainly the content of a work of the three authors which is a chapter of the Ph.D. thesis [21]

For the subset S of H, $\overline{\mathrm{co}}\, S$ stands for the closed convex hull of S, and $\sigma(\cdot, S)$ represents the support function of S, that is, for all $\xi \in H$,

$$\sigma(\xi, S) := \sup_{y \in S} \langle \xi, y \rangle.$$

The Clarke normal cone $N(S; x)$ or $N_S(x)$ of S at $x \in S$ is defined by

$$N_S(x) = \{\xi \in H : \langle \xi, v \rangle \le 0, \forall v \in T(S; x)\},$$

where $T(S; x)$ or $T_S(x)$ (see [10]) is the Clarke tangent cone of S at $x \in S$ defined as follows:

$$v \in T(S; x) \Leftrightarrow \begin{cases} \forall \varepsilon > 0, \exists \delta > 0 \text{ such that} \\ \\ \forall x' \in B(x, \delta) \cap S, \forall t \in]0, \delta[, (x' + tB(v, \varepsilon)) \cap S \ne \emptyset. \end{cases}$$

Equivalently, $v \in T(S; x)$ if and only if for any sequence $(x_n)_n$ of S converging to x and any sequence of positive reals $(t_n)_n$ converging to 0, there exists a sequence $(v_n)_n$ in H converging to v such that

$$x_n + t_n v_n \in S \quad \text{for all } n \in \mathbb{N}.$$

We put $N(S; x) = \emptyset$, whenever $x \notin S$. We typically denote by $f : H \to \mathbb{R} \cup \{+\infty\}$ a proper function (that is, f is finite at least at one point). The Clarke subdifferential $\partial f(x)$ of f at a point $x \in \operatorname{dom} f := \{x' \in H : f(x') < +\infty\}$ (i.e., $f(x)$ is finite) is defined by

$$\partial f(x) = \Big\{\xi \in H : (\xi, -1) \in N_{\operatorname{epi} f}\Big(\big(x, f(x)\big)\Big)\Big\},$$

where $\operatorname{epi} f$ denotes the epigraph of f, that is,

$$\operatorname{epi} f = \{(u, r) \in H \times \mathbb{R} : f(u) \le r\}.$$

We also put $\partial f(x) = \emptyset$ if f is not finite at $x \in H$. We denote by $\operatorname{Dom} \partial f$ the (effective) domain of the multimapping $\partial f : H \rightrightarrows H$, that is,

$$\operatorname{Dom} \partial f := \{x \in H : \partial f(x) \ne \emptyset\}.$$

If ψ_S denotes the indicator function of the set S, that is, $\psi_S(x) = 0$ if $x \in S$ and $\psi_S(x) = +\infty$ otherwise, then

$$\partial \psi_S(x) = N(S; x) \quad \text{for all } x \in H.$$

The Clarke subdifferential $\partial f(x)$ of a locally Lipschitz function f at x has also the other useful description

$$\partial f(x) = \{\xi \in H : \langle \xi, v \rangle \le f^o(u; v), \forall v \in H\},$$

where

$$f^o(x;v) := \limsup_{(x',t)\to(x,0^+)} \frac{f(x'+tv)-f(x')}{t}.$$

The above function $f^o(x;\cdot)$ is called the Clarke directional derivative of f at x. The Clarke normal cone is known ([10]) to be related to the Clarke subdifferential of the distance function through the equality

$$N(S;x) = \mathrm{cl}_w(\mathbb{R}_+ \partial d_S(x)) \text{ for all } x \in S,$$

where $\mathbb{R}_+ := [0,\infty[$ and cl_w denotes the closure with respect to the weak topology of H. Further,

$$\partial d_S(x) \subset N(S;x) \cap \mathbb{B} \quad \text{for all } x \in S. \tag{1.1}$$

The concept of Fréchet subdifferential will also be needed. A vector $\xi \in H$ is said to be in the Fréchet subdifferential $\partial_F f(x)$ of f at x (see, e.g., [17]) provided that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x' \in B(x,\delta)$ we have

$$\langle \xi, x'-x\rangle \le f(x') - f(x) + \varepsilon\|x'-x\|.$$

It is known that we always have the inclusion

$$\partial_F f(x) \subset \partial f(x).$$

The Fréchet normal cone $N^F(S;x)$ or $N_S^F(x)$ of S at $x \in S$ is given by

$$N^F(S;x) = \partial_F \psi_S(x),$$

so the following inclusion always holds true

$$N^F(S;x) \subset N(S;x) \quad \text{for all } x \in S.$$

On the other hand, the Fréchet normal cone is also related (see, e.g., [17]) to the Fréchet subdifferential of the distance function since the following relations hold true for all $x \in S$

$$N^F(S;x) = \mathbb{R}_+ \partial_F d_S(x)$$

and

$$\partial_F d_S(x) = N^F(S;x) \cap \mathbb{B}. \tag{1.2}$$

Another important property is that, whenever $y \in \mathrm{Proj}\,_S(x)$, one has

$$x - y \in N^F(S;y), \quad \text{hence also} \quad x-y \in N(S;y), \tag{1.3}$$

since the proximal normal cone

$$N^p(S;y) := \mathbb{R}_+\big(\mathrm{Proj}_S^{-1}(y) - y\big) \tag{1.4}$$

is known to be included in the Fréchet one.

We now recall the definition of subsmooth sets introduced in [1] (see also [12] for various variants).

Definition 1.1 *Let S be a closed subset of H. We say that S is subsmooth at $x_0 \in S$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\langle \xi_2 - \xi_1, x_2 - x_1 \rangle \geq -\varepsilon \|x_2 - x_1\|, \tag{1.5}$$

whenever $x_1, x_2 \in B(x_0, \delta) \cap S$ and $\xi_i \in N(S; x_i) \cap \mathbb{B}$. The set S is subsmooth, if it is subsmooth at each point of S. We further say that S is uniformly subsmooth, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that (1.5) holds for all $x_1, x_2 \in S$ satisfying $\|x_1 - x_2\| < \delta$ and all $\xi_i \in N(S; x_i) \cap \mathbb{B}$.

Since $0 \in N(S; x_2) \cap \mathbb{B}$, the set S is clearly subsmooth at $x_0 \in S$ (resp. uniformly subsmooth) provided that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\langle \xi_1, x_2 - x_1 \rangle \leq \varepsilon \|x_2 - x_1\| \tag{1.6}$$

for all $x_1, x_2 \in B(x_0, \delta) \cap S$ and $\xi_1 \in N(S; x_1) \cap \mathbb{B}$ (resp. for all $x_1, x_2 \in S$ satisfying $\|x_1 - x_2\| < \delta$ and all $\xi_1 \in N(S; x_1) \cap \mathbb{B}$).

Let us make the connection between the subsmoothness property and other geometrical concepts.

It has been recognized that the concept of prox-regularity of a set at a point, developed by R. A. Poliquin, R. T. Rockafellar and L. Thibault [24], plays an important role in variational analysis. In [1] it is proved that if a closed set S of H is uniformly prox-regular (respectively, prox-regular at $x_0 \in S$), then it is uniformly subsmooth (respectively, subsmooth at x_0). In order to give simple examples where the converse implication fails, let us recall the concept of subsmooth function.

Definition 1.2 *A function $f : H \to \mathbb{R} \cup \{+\infty\}$ Lipschitz near $x_0 \in \operatorname{dom} f$ is called subsmooth at x_0, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, x' \in B(x_0, \delta)$ and $\xi \in \partial f(x)$*

$$f(x') \geq f(x) + \langle \xi, x' - x \rangle + \varepsilon \|x' - x\|.$$

Obviously, every function f which is of class $\mathcal{C}^1$ on an open set $U \subset H$ is subsmooth at each point in U.

Proposition 1.1 *[1] Let $f : H \to \mathbb{R}$ be a function which is locally Lipschitz near $x_0 \in H$. Then f is subsmooth at x_0 if only if the set $\operatorname{epi} f$ is subsmooth at $(x_0, f(x_0))$.*

Proposition 1.2 *[1] Let S be a closed subset of H and $x_0 \in S$. Then the following assertions hold:*
(a) If S is subsmooth at x_0, then it is normally Fréchet regular at x_0, that is,

$$N^F(S; x_0) = N(S; x_0).$$

(b) If S is prox-regular at x_0, then it is normally regular at x_0, that is,

$$N^p(S; x_0) = N^F(S; x_0) = N(S; x_0).$$

Consider now the $\mathcal{C}^1$ function on $\mathbb{R}$ defined by $f(x) = -x^{5/3}$ for all $x \in \mathbb{R}$. With $E_f := \operatorname{epi} f$ and $z_0 := (0,0)$, it is not difficult to see that

$$N^F(E_f; z_0) = \{(x,r) \in \mathbb{R}^2 : x = 0 \text{ and } r \leq 0\} \quad \text{and} \quad N^p(E_f; z_0) = \{(0,0)\},$$

then $N^F(E_f; z_0) \neq N^p(E_f; z_0)$. Consequently, the set E_f is subsmooth by Proposition 1.1 while it is not prox-regular at z_0 according to the assertion (b) in Proposition1.2.

It is worth mentioning that any C^1 (resp. C^2) submanifold is subsmooth (resp. prox-regular). Besides this class and that in Proposition 1.1, there are plenty of nonsmooth subsmooth sets. For example, it is easily seen that the nonsmooth set $S := \{(x,r) \in \mathbb{R}^2 : x \geq 0,\, r \geq -x^{5/3}\}$ in $\mathbb{R}^2$ is subsmooth; note also that it is not prox-regular at $(0,0)$.

The following subdifferential regularity of the distance function also holds true for subsmooth sets:

Proposition 1.3 *[12, 21] If a closed set S of H is subsmooth at $x_0 \in S$, then*

$$\partial d_S(x_0) = \partial_F d_S(x_0).$$

We now introduce the definition of *equi-uniform subsmoothness for a family of sets*. The notion will be used in the proof of various results.

Definition 1.3 *Let $\big(S(q)\big)_{q \in Q}$ be a family of closed sets of H with parameter $q \in Q$. This family is called equi-uniformly subsmooth, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for each $q \in Q$, the inequality (1.5) (or equivalently (1.6)) holds for all $x_1, x_2 \in S(q)$ satisfying $\|x_1 - x_2\| < \delta$ and all $\xi_i \in N\big(S(q); x_i\big) \cap \mathbb{B}$ (resp. $\xi_1 \in N\big(S(q); x_1\big) \cap \mathbb{B}$).*

The following lemma related to subsmooth sets will be used in the next proposition.

Lemma 1.1 *Let E be a metric space and let $(S(q))_{q \in E}$ be a family of nonempty closed sets of H which is equi-uniformly subsmooth and let a real $\eta \geq 0$. Let $Q \subset E$ and $q_0 \in \operatorname{cl} Q$. Then the following hold:*

(a) For all $(q,x) \in \operatorname{gph} S$ we have $\eta \partial d_{S(q)}(x) \subset \eta \mathbb{B}$;

(b) For any net $(q_j)_{j \in J}$ in Q converging to q_0, any net $(x_j)_{j \in J}$ converging to $x \in S(q_0)$ in $(H, \|\cdot\|)$ with $x_j \in S(q_j)$ and $d_{S(q_j)}(y) \underset{j \in J}{\to} 0$ for every $y \in S(q_0)$, and any net $(\xi_j)_{j \in J}$ converging weakly to ξ in $(H, w(H,H))$ with $\xi_j \in \eta \partial d_{S(q_j)}(x_j)$, we have $\xi \in \eta \partial d_{S(q_0)}(x)$.

Proof. The assertion (a) being due to (1.1), we have to show (b). Of course, we may suppose that $\eta > 0$. Take any real $\varepsilon > 0$. By Definition of equi-uniform subsmoothness choose $\delta > 0$ such that for all $q \in E$, $x', x'' \in S(q)$ with $\|x' - x''\| < \delta$ and all $\xi' \in N(S(q); x') \cap \mathbb{B}$ and $\xi'' \in N(S(q); x'') \cap \mathbb{B}$

$$\langle \xi' - \xi'', x' - x'' \rangle \geq -\varepsilon \|x' - x''\|. \tag{1.7}$$

Fix any nets $(q_j)_{j \in J}$ in Q converging to q_0, $(x_j)_{j \in J}$ converging strongly to $x \in S(q_0)$ in H with $x_j \in S(q_j)$ and $d_{S(q_j)}(y) \underset{j \in J}{\to} 0$ for every $y \in S(q_0)$, where $(J, \preccurlyeq)$ is a directed preordered set. Fix also any net $(\xi_j)_{j \in J}$ converging weakly to ξ in H such that $\xi_j \in \eta \partial d_{S(q_j)}(x_j)$. Since $x_j \in S(q_j)$, the latter inclusion means $\eta^{-1}\xi_j \in N_{S(q_j)}(x_j) \cap \mathbb{B}$ for all $j \in J$ (see (1.2) and Proposition 1.3). Fix $y \in B(x, \frac{\delta}{2}) \cap S(q_0)$. For each $n \in \mathbb{N}$ and each $j \in J$, choose some $y_{j,n} \in S(q_j)$ such that

$$\|y_{j,n} - y\| \leq d_{S(q_j)}(y) + \frac{1}{n}.$$

Endowing $J \times \mathbb{N}$ with the product preorder which is obviously directed, the family $(y_{j,n})_{(j,n) \in J \times \mathbb{N}}$ is a net in H. Since

$$d_{S(q_j)}(y) + \frac{1}{n} \underset{(j,n) \in J \times \mathbb{N}}{\longrightarrow} 0,$$

we have $\|y_{j,n} - y\| \underset{(j,n) \in J \times \mathbb{N}}{\longrightarrow} 0$, that is, $y_{j,n} \underset{(j,n) \in J \times \mathbb{N}}{\longrightarrow} y$ strongly in H, and hence there exists $j_0 \in J$ and $n_0 \in \mathbb{N}$ such that for all $(j,n) \in J \times \mathbb{N}$ with $j \succcurlyeq j_0$ and $n \geq n_0$ we have $y_{j,n} \in B(x, \frac{\delta}{2})$. Put $x_{j,n} := x_j$ for all $(j,n) \in J \times \mathbb{N}$. Obviously $x_{j,n} \underset{(j,n) \in J \times \mathbb{N}}{\longrightarrow} x$ strongly in H (because $x_j \underset{j \in J}{\to} x$). So, we may also suppose that $x_{j,n} \in B(x, \frac{\delta}{2})$ for all $(j,n) \in J \times \mathbb{N}$, with $j \succcurlyeq j_0$ and $n \geq n_0$. Thus, for all $(j,n) \in J \times \mathbb{N}$ with $j \succcurlyeq j_0$ and $n \geq n_0$ we have

$$\|y_{j,n} - x\| < \frac{\delta}{2} \text{ and } \|x_{j,n} - x\| < \frac{\delta}{2}.$$

Set $\xi_{j,n} := \xi_j$ and $q_{j,n} := q_j$ for all $(j,n) \in J \times \mathbb{N}$. The net $(q_{j,n})_{(j,n) \in J \times \mathbb{N}}$ converges to q_0 and the net $\xi_{j,n})_{(j,n) \in J \times \mathbb{N}}$ converges weakly to ξ in H and $\eta^{-1}\xi_{j,n} \in N_{S(q_{j,n})}(x_{j,n}) \cap \mathbb{B}$. Thanks to the latter inequalities above, for all $(j,n) \in J \times \mathbb{N}$ with $j \succcurlyeq j_0$ and $n \geq n_0$ we have $\|y_{j,n} - x_{j,n}\| < \delta$ with $y_{j,n}, x_{j,n} \in S(q_{j,n})$, and hence according to (1.7))

$$\langle 0 - \eta^{-1}\xi_{j,n}, y_{j,n} - x_{j,n} \rangle \geq -\varepsilon \|y_{j,n} - x_{j,n}\|$$

or equivalently

$$\langle \eta^{-1}\xi_{j,n}, y_{j,n} - x_{j,n} \rangle \leq \varepsilon \|y_{j,n} - x_{j,n}\|.$$

Since the net $(\eta^{-1}\xi_{j,n})_{(j,n) \in J \times \mathbb{N}}$ is bounded (by the real number 1), we may pass to the limit to obtain

$$\langle \eta^{-1}\xi, y - x \rangle \leq \varepsilon \|y - x\|$$

for all $y \in B(x, \frac{\delta}{2}) \cap S(q_0)$ and hence $\eta^{-1}\xi \in N^F_{S(q_0)}(x)$. Further, $\eta^{-1}\xi_{j,n} \in \mathbb{B}$ for all $(j,n) \in J \times \mathbb{N}$ and this ensures $\eta^{-1}\xi \in \mathbb{B}$. Thus, $\eta^{-1}\xi \in N^F_{S(q_0)}(x) \cap \mathbb{B}$, so (1.2) gives $\eta^{-1}\xi \in \partial_F d_{S(q_0)}(x) \subset \partial d_{S(q_0)}(x)$. The proof is then complete. $\square$

Through Lemma 1.1 we can establish the following partial upper semicontinuity property.

Proposition 1.4 *Let $\{C(t,x):\ (t,x)\in[0,T]\times H\}$ be a family of nonempty closed sets of H which is equi-uniformly subsmooth and let a real $\eta\geq 0$. Assume that there exist a real constant $L\geq 0$ and a continuous function $\vartheta:[0,T]\to\mathbb{R}$ such that, for any $x,x',y,y'\in H$ and $s,t\in[0,T]$*

$$|d\big(y,C(t,x)\big)-d\big(y',C(s,x')\big)|\leq\|y-y'\|+|\vartheta(t)-\vartheta(s)|+L\|x-x'\|.$$

Then the following assertions hold:

(a) For all $(s,x,y)\in\operatorname{gph}C$ we have $\eta\partial d_{C(s,x)}(y)\subset\eta\mathbb{B}$;

(b) For any sequence $(s_n)_n$ in $[0,T]$ converging to s, any sequence $(x_n)_n$ converging to x, any sequence $(y_n)_n$ converging to $y\in C(s,x)$ with $y_n\in C(s_n,x_n)$, and any $\xi\in H$, we have

$$\limsup_{n\to\infty}\sigma\big(\xi,\eta\partial d_{C(s_n,x_n)}(y_n)\big)\leq\sigma\big(\xi,\eta\partial d_{C(s,x)}(y)\big).$$

Proof. The proof will be an application of Lemma 1.1 above. Since (a) is obvious, we only have to prove (b). Let $(s_n)_n$, $(x_n)_n$ and $(y_n)_n$ be as in the statement. Extracting a subsequence if necessary, we may suppose that

$$\limsup_{n\to\infty}\sigma\big(\xi,\eta\partial d_{C(s_n,x_n)}(y_n)\big)=\lim_{n\to\infty}\sigma\big(\xi,\eta\partial d_{C(s_n,x_n)}(y_n)\big).$$

For any n, the weak compactness of $\eta\partial d_{C(s_n,x_n)}(y_n)$ ensures the existence of some $\zeta_n\in\eta\partial d_{C(s_n,x_n)}(y_n)$ such that

$$\langle\xi,\zeta_n\rangle=\sigma\big(\xi,\eta\partial d_{C(s_n,x_n)}(y_n)\big).$$

Since $\|\zeta_n\|\leq\eta$ by (a), a subsequence of $(\zeta_n)_n$ (that we do not relabel) converges weakly to some ζ in H. It results that

$$\langle\xi,\zeta\rangle=\limsup_{n\to\infty}\sigma\big(\xi,\eta\partial d_{C(s_n,x_n)}(y_n)\big).\tag{1.8}$$

Now, observe, for each $z\in C(s,x)$, that

$$0\leq d\big(z,C(s_n,x_n)\big)\leq d\big(z,C(s,x)\big)+|\vartheta(s_n)-\vartheta(s)|+L\|x_n-x\|.$$

Since $(x_n)_n$ and $(s_n)_n$ converges to x and s respectively, it follows that $d\big(z,C(s_n,x_n)\big)$ converge to 0. We then apply Lemma 1.1 to obtain that $\zeta\in\eta\partial d_{C(s,x)}(y)$. The latter inclusion combined with (1.8) yields

$$\limsup_{n\to\infty}\sigma\big(\xi,\eta\partial d_{C(s_n,x_n)}(y_n)\big)\leq\sigma\big(\xi,\eta\partial d_{C(s,x)}(y)\big),$$

which completes the proof. $\square$

2 Quasi-variational inequality and subsmoothness property

We show in this section under suitable assumptions that there always exists a solution for quasi-variational differential inequality/inclusion governed by subsmooth sets.

We shall be dealing with two multimappings $G : [0,T] \times H \rightrightarrows H$ with *nonempty closed convex values* and $C : [0,T] \times H \rightrightarrows H$ with nonempty values. In Theorem 2.1 they are required to satisfy the following assumptions:

($\mathcal{H}_1$) The multimapping G is scalarly upper semicontinuous with respect to both variables (that is, for each $\xi \in H$ the function $(t,x) \to \sigma(\xi, G(t,x))$ is upper semicontinuous), and for some real $\alpha \geq 0$

$$d\big(0, G(t,x)\big) \leq \alpha$$

for all $t \in [0,T]$ and $x \in H$;

($\mathcal{H}_2$) For all $t \in [0,T]$ and $x \in H$, the sets $C(t,x)$ are nonempty and equi-uniformly subsmooth (with parameter $(t,x) \in [0,T] \times H$);

($\mathcal{H}_3$) There are real constants $L_1 \geq 0$, $L_2 \in [0,1[$ such that, for all $t,s \in [0,T]$ and $x,x',y,y' \in H$

$$\big|d\big(y, C(t,x)\big) - d\big(y', C(s,x')\big)\big| \leq \|y - y'\| + L_1|t-s| + L_2\|x - x'\|.$$

($\mathcal{H}_4$) For any bounded subset $A \subset H$, the set $C([0,T] \times A)$ is relatively ball-compact, that is, the intersection of $C([0,T] \times A)$ with any closed ball of H is relatively compact in H.

Of course the inequality condition in ($\mathcal{H}_3$) is equivalent to

$$\big|d\big(y, C(t,x)\big) - d\big(y, C(s,x')\big)\big| \leq L_1|t-s| + L_2\|x - x'\|$$

for all $t,s \in [0,T]$ and $x,x',y \in H$.

Theorem 2.1 *Assume that H is a Hilbert space and that $(\mathcal{H}_1), \cdots, (\mathcal{H}_4)$ hold. Then, for any $u_0 \in H$ with $u_0 \in C(0,u_0)$, there exists a Lipschitz continuous solution $u : [0,T] \to H$ of the differential inclusion*

$$(\mathcal{D}) \quad \begin{cases} \dot{u}(t) \in -N_{C\big(t,u(t)\big)}\big(u(t)\big) + G\big(t,u(t)\big) \quad a.e \;\; t \in [0,T], \\ \\ u(t) \in C\big(t,u(t)\big) \; \forall t \in [0,T], \; u(0) = u_0, \end{cases}$$

with $\|\dot{u}(t)\| \leq \frac{L_1+2\alpha}{1-L_2}$ a.e. $t \in [0,T]$.

Proof. For each integer $n \geq 1$, we consider the partition of $[0,T]$ with the points

$$t_k^n = k\frac{T}{n},\ k = 0, 1, \cdots, n.$$

For each $(t,x) \in [0,T] \times H$ denote by $g(t,x)$ the element of minimal norm of the closed convex set $G(t,x)$ of H, that is,

$$g(t,x) = \text{proj}_{\,G(t,x)}(0).$$

Put $x_0^n := u_0 \in C(t_0^n, u_0)$.

Step 1. We construct $x_0^n, x_1^n, \cdots, x_n^n$ in H such that for each $k = 0, 1, \cdots, n-1$, the following inclusions hold

$$x_{k+1}^n \in C(t_{k+1}^n, x_k^n) \tag{2.1}$$

$$x_k^n + \frac{T}{n}g(t_k^n, x_k^n) - x_{k+1}^n \in N_{C(t_{k+1}^n, x_k^n)}(x_{k+1}^n), \tag{2.2}$$

along with the inequality $\|x_1^n - x_0^n\| \leq (L_1 + 2\alpha)\frac{T}{n}$, and for $k = 1, \cdots, n-1$

$$\|x_{k+1}^n - x_k^n\| \leq (L_1 + 2\alpha)\frac{T}{n} + L_2\|x_k^n - x_{k-1}^n\|. \tag{2.3}$$

The ball-compactness of $C(t_1^n, x_0^n)$ ensures that we can choose

$$x_1^n \in \text{Proj}_{\,C(t_1^n, x_0^n)}\Big(x_0^n + \frac{T}{n}g(t_0^n, x_0^n)\Big)$$

and hence

$$x_1^n \in C(t_1^n, x_0^n)$$

$$x_0^n + \frac{T}{n}g(t_0^n, x_0^n) - x_1^n \in N_{C(t_1^n, x_0^n)}(x_1^n) \text{ by (1.3).}$$

On the other hand, using $\|g(t_0^n, x_0^n)\| \leq \alpha$, in view of hypothesis $(\mathcal{H}_1)$ we have

$$\begin{aligned}
\|x_1^n - x_0^n\| &\leq \Big\|x_1^n - \big(x_0^n + \frac{T}{n}g(t_0^n, x_0^n)\big)\Big\| + \Big\|\frac{T}{n}g(t_0^n, x_0^n)\Big\| \\
&= d\Big(x_0^n + \frac{T}{n}g(t_0^n, x_0^n), C(t_1^n, x_0^n)\Big) + \Big\|\frac{T}{n}g(t_0^n, x_0^n)\Big\| \\
&\leq d\Big(x_0^n + \frac{T}{n}g(t_0^n, x_0^n), C(t_0^n, x_0^n)\Big) + L_1|t_1^n - t_0^n| + \Big\|\frac{T}{n}g(t_0^n, x_0^n)\Big\| \\
&\leq 2\Big\|\frac{T}{n}g(t_0^n, x_0^n)\Big\| + L_1\frac{T}{n} \\
&\leq \Big(L_1 + 2\alpha\Big)\frac{T}{n}.
\end{aligned} \tag{2.4}$$

Now, suppose that, for $0, 1, \cdots, k$, with $k \leq n-1$ the points $x_0^n, x_1^n, \cdots, x_k^n$ have been constructed, so that properties (2.1), (2.2) and (2.3) hold true. Since $C(t_{k+1}^n, x_k^n)$ is ball-compact, then we can find

$$x_{k+1}^n \in \text{Proj}_{\,C(t_{k+1}^n, x_k^n)}\Big(x_k^n + \frac{T}{n}g(t_k^n, x_k^n)\Big),$$

and therefore,

$$x^n_{k+1} \in C(t^n_{k+1}, x^n_k),$$

$$x^n_k + \frac{T}{n} g(t^n_k, x^n_k) - x^n_{k+1} \in N_{C(t^n_{k+1}, x^n_k)}(x^n_{k+1}) \text{ by (1.3).}$$

By $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$, we get

$$\begin{aligned}
\|x^n_{k+1} - x^n_k\| &\leq \Big\| x^n_{k+1} - \big(x^n_k + \frac{T}{n} g(t^n_k, x^n_k)\big)\Big\| + \Big\|\frac{T}{n} g(t^n_k, x^n_k)\Big\| \\
&= d\Big(x^n_k + \frac{T}{n} g(t^n_k, x^n_k), C(t^n_{k+1}, x^n_k)\Big) + \Big\|\frac{T}{n} g(t^n_k, x^n_k)\Big\| \\
&\leq d\Big(x^n_k + \frac{T}{n} g(t^n_k, x^n_k), C(t^n_k, x^n_{k-1})\Big) + \Big\|\frac{T}{n} g(t^n_k, x^n_k)\Big\| \\
&\quad + L_1|t^n_{k+1} - t^n_k| + L_2\|x^n_k - x^n_{k-1}\| \\
&\leq 2\alpha\frac{T}{n} + L_1\frac{T}{n} + L_2\|x^n_k - x^n_{k-1}\|.
\end{aligned}$$

The finite sequence $x^n_0, x^n_1 \cdots, x^n_n$ satisfying (2.1), (2.2) and (2.3) is then contructed by induction.

Fix any $k = 1, \cdots, n-1$. We observe that

$$\begin{aligned}
\|x^n_{k+1} - x^n_k\| &\leq 2\alpha\frac{T}{n} + L_1\frac{T}{n} + L_2\|x^n_k - x^n_{k-1}\| \\
&\leq 2\alpha\frac{T}{n} + L_1\frac{T}{n} + L_2\Big(2\alpha\frac{T}{n} + L_1\frac{T}{n} + L_2\|x^n_{k-1} - x^n_{k-2}\|\Big) \\
&= 2\alpha\frac{T}{n}(1 + L_2) + L_1\frac{T}{n}(1 + L_2) + L_2^2\|x^n_{k-1} - x^n_{k-2}\|,
\end{aligned}$$

thus, we deduce that

$$\begin{aligned}
\|x^n_{k+1} - x^n_k\| &\leq 2\alpha\frac{T}{n}(1 + L_2 + L_2^2 + \cdots + L_2^{k-1}) \\
&\quad + L_1\frac{T}{n}(1 + L_2 + {L_2}^2 + \cdots + L_2^{k-1}) + L_2^k\|x^n_1 - x^n_0\| \\
&= \frac{T}{n}(2\alpha + L_1)(1 + L_2 + L_2^2 + \cdots + L_2^{k-1}) + L_2^k\|x^n_1 - x^n_0\|.
\end{aligned}$$

From this and from (2.4) it follows from that

$$\|x^n_{k+1} - x^n_k\| \leq (2\alpha + L_1)\big(1 + L_2 + L_2^2 + \cdots + L_2^k\big)\frac{T}{n},$$

and since $0 \leq L_2 < 1$, it results that

$$\|x^n_{k+1} - x^n_k\| \leq \frac{2\alpha + L_1}{1 - L_2}\frac{T}{n}, \tag{2.5}$$

and the latter inequality still holds true for $k=0$ according to (2.4). Further for $k=0,\cdots,n-1$,

$$\begin{aligned}\|x_{k+1}^n\| &\le \|x_{k+1}^n - x_k^n\| + \|x_k^n - x_{k-1}^n\| + \cdots + \|x_1^n - x_0^n\| + \|x_0^n\| \\ &\le \|u_0\| + \frac{2\alpha+L_1}{1-L_2}(k+1)\frac{T}{n} \\ &\le \|u_0\| + \frac{2\alpha+L_1}{1-L_2}T =: \beta. \end{aligned} \tag{2.6}$$

Step 2. Construction of $u_n(\cdot)$.
For any $t\in[t_k^n, t_{k+1}^n]$ with $k=0,1,\cdots,n-1$, put

$$u_n(t) := \frac{t_{k+1}^n - t}{t_{k+1}^n - t_k^n}x_k^n + \frac{t-t_k^n}{t_{k+1}^n - t_k^n}x_{k+1}^n.$$

Thus, for almost all $t\in[t_k^n, t_{k+1}^n]$,

$$\dot u_n(t) = -\frac{x_k^n}{t_{k+1}^n - t_k^n} + \frac{x_{k+1}^n}{t_{k+1}^n - t_k^n} = -\frac{n}{T}(x_k^n - x_{k+1}^n).$$

By construction, (2.1), (2.2), (2.3) and the latter equalities give

$$u_n(t_{k+1}^n) \in C\big(t_{k+1}^n, u_n(t_k^n)\big) \tag{2.7}$$

$$-\dot u_n(t) \in N_{C\left(t_{k+1}^n, u_n(t_k^n)\right)}\big(u_n(t_{k+1}^n)\big) - g\big(t_k^n, u_n(t_k^n)\big) \text{ a.e. } t\in[t_k^n, t_{k+1}^n[\tag{2.8}$$

with (by (2.5))

$$\|\dot u_n(t)\| = \frac{n}{T}\|x_k^n - x_{k+1}^n\| \le \frac{L_1+2\alpha}{1-L_2} =: M. \tag{2.9}$$

Put

$$\delta_n(t) := \begin{cases} t_k^n & \text{if} \quad t\in[t_k^n, t_{k+1}^n[\\ t_{n-1}^n & \text{if} \quad t=T, \end{cases}$$

and

$$\theta_n(t) := \begin{cases} t_{k+1}^n & \text{if} \quad t\in[t_k^n, t_{k+1}^n[\\ T & \text{if} \quad t=T. \end{cases}$$

Observe that for each $t\in[0,T]$, choosing k such that $t\in[t_k^n, t_{k+1}^n[$ if $t<T$ and $k=n-1$ if $t=T$, we have

$$|\delta_n(t)-t| \le |t_{k+1}^n - t_k^n| = \frac{T}{n}, \text{ so } \delta_n(t)\to t \text{ as } n\to+\infty,$$

and similarly $\theta_n(t)\to t$ as $n\to+\infty$. Further, the definitions of $\delta_n(\cdot)$ and $\theta_n(\cdot)$ combined with (2.7) and (2.8) yield

$$u_n\big(\theta_n(t)\big) \in C\Big(\theta_n(t), u_n\big(\delta_n(t)\big)\Big) \quad \text{for all } t\in[0,T] \tag{2.10}$$

$$-\dot{u}_n(t) \in N_{C\left(\theta_n(t), u_n\left(\delta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big) - g\Big(\delta_n(t), u_n\big(\delta_n(t)\big)\Big) \text{ a.e } t \in [0,T]. \tag{2.11}$$

Step 3. Convergence of a subsequence of $(u_n(\cdot))$ to some absolutely continuous mapping $u(\cdot)$.
Fix any $t \in [0,T]$ and consider, for any infinite subset $N \subset \mathbb{N}$, the sequence $(u_n(t))_{n\in N}$. It follows from (2.6) and (2.10) that

$$u_n\big(\theta_n(t)\big) \in C\Big(\theta_n(t), u_n\big(\delta_n(t)\big)\Big) \cap \beta\mathbb{B},$$

which implies that $u_n(\theta_n(t)) \in C([0,T] \times \beta\mathbb{B}) \cap \beta\mathbb{B}$. By $(\mathcal{H}_4)$ the sequence $(u_n(\theta_n(t)))$ is relatively compact, so there is an infinite subset $N_0 \subset N$ such that $(u_n(\theta_n(t)))_{n\in N_0}$ converges to some vector $l(t) \in H$. Putting $h_n(t) := u_n(\theta_n(t)) - u_n(t)$ for all $n \in N_0$, by (2.9), we obtain

$$\|h_n(t)\| \le \int_t^{\theta_n(t)} \|\dot{u}_n(s)\| ds \le M(\theta_n(t) - t) \underset{n\to\infty}{\longrightarrow} 0.$$

Then, $(u_n(t))_{n\in N_0}$ converges to $l(t)$, thus the set $\{u_n(t) : n \in \mathbb{N}\}$ is relatively compact in H. The sequence $(u_n)_{n\in\mathbb{N}}$ being in addition equicontinuous according to (2.9), this sequence $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{C}_H(0,T)$, so we can extract a subsequence of $(u_n)_{n\in\mathbb{N}}$ (that we do not relabel) which converges uniformly to some mapping u on $[0,T]$. By the inequality (2.9) again there is a subsequence of $(\dot{u}_n)_{n\in\mathbb{N}}$ (that we do not relabel) which converges weakly in $\mathrm{L}^2_H(0,T)$ to a mapping $w \in \mathrm{L}^2_H(0,T)$ with $\|w(t)\| \le M$ a.e. $t \in [0,T]$. Fixing $t \in [0,T]$ and taking any $\xi \in H$, the above weak convergence in $\mathrm{L}^2_H(0,T)$ yields

$$\lim_{n\to\infty} \int_0^T \langle \mathbb{1}_{[0,t]}(s)\xi, \dot{u}_n(s)\rangle ds = \int_0^T \langle \mathbb{1}_{[0,t]}(s)\xi, w(s)\rangle ds,$$

or equivalently

$$\lim_{n\to\infty} \langle \xi, u_0 + \int_0^t \dot{u}_n(s)\, ds\rangle = \langle \xi, u_0 + \int_0^t w(s)\, ds\rangle.$$

This means, for each $t \in [0,T]$, that $u_n(t) \underset{n\to\infty}{\longrightarrow} u_0 + \int_0^t w(s)\, ds$ weakly in H. Since the sequence $(u_n(t))_{n\in\mathbb{N}}$ also converges strongly to $u(t)$ in H, it ensures that $u(t) = u_0 + \int_0^t w(s)\, ds$, so the mapping $u(\cdot)$ is absolutely continuous on $[0,T]$ with $\dot{u} = w$. The mapping $u(\cdot)$ is even Lipschitz on $[0,T]$ with M as a Lipschitz constant therein.

Step 4. We show now that $u(\cdot)$ is a solution of $(\mathcal{D})$.
Let $z_n(t) := g(\delta_n(t), u_n(\delta_n(t)))$ for all $t \in [0,T]$. Since

$$\|g(\delta_n(t), u_n(\delta_n(t)))\| \le \alpha \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [0,T],$$

we may suppose (taking a subsequence if necessary) that the sequence $(z_n(\cdot))$ converges weakly in $L^2_H(0,T)$ to a mapping $z(\cdot) \in \mathrm{L}^2_H(0,T)$ with $\|z(t)\| \le \alpha$ a.e $t \in [0,T]$.

For all $t \in [0,T]$ we have $u(t) \in C(t,u(t))$. Indeed, by $(\mathcal{H}_3)$ and (2.9)

$$
\begin{aligned}
&d\Big(u_n(t), C\big(t,u(t)\big)\Big)\\
&\qquad \le \|u_n(t) - u_n\big(\theta_n(t)\big)\| + L_1|t-\theta_n(t)| + L_2\|u(t) - u_n\big(\delta_n(t)\big)\|\\
&\qquad \le (M+L_1)|t-\theta_n(t)| + L_2 M|\delta_n(t)-t| + L_2\|u(t)-u_n(t)\|,
\end{aligned}
$$

then

$$
d(u_n(t), C(t,u(t))) \underset{n\to\infty}{\longrightarrow} 0, \text{ so } d(u(t),C(t,u(t))) = 0 \text{ and } u(t) \in C(t,u(t)).
$$

Further, from the inequality $\|\dot{u}_n(t) - z_n(t)\| \le M + \alpha =: \gamma$ a.e. and from the inclusion (2.11) it follows for a.e. $t \in [0,T]$ that, according to (1.2) and Proposition 1.3,

$$
-\dot{u}_n(t) + z_n(t) \in N_{C\left(\theta_n(t), u_n\left(\delta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big) \bigcap \gamma\mathbb{B} \tag{2.12}
$$

$$
= \gamma\partial d_{C\left(\theta_n(t), u_n\left(\delta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big), \tag{2.13}
$$

$$
z_n(t) \in G\Big(\delta_n(t), u_n\big(\delta_n(t)\big)\Big). \tag{2.14}
$$

Since $(-\dot{u}_n + z_n, z_n)_n$ converges weakly in $\mathrm{L}^2_{H\times H}(0,T)$ to $(-\dot{u}+z, z)$, so by Mazur theorem there are

$$
\xi_n \in \mathrm{co}\{-\dot{u}_q + z_q : q \ge n\} \text{ and } \zeta_n \in \mathrm{co}\{z_q : q \ge n\} \tag{2.15}
$$

such that $(\xi_n, \zeta_n)_n$ converges strongly in $\mathrm{L}^2_{H\times H}(0,T)$ to $(-\dot{u}+z,z)$. Extracting a subsequence if necessary we suppose that $(\xi_n(\cdot), \zeta_n(\cdot))_n$ converges a.e. to $(-\dot{u}(\cdot) + z(\cdot), z(\cdot))$. Then, there is a Lebesgue negligible set $S \subset [0,T]$ such that, for every $t \in [0,T]\backslash S$, on one hand $(\xi_n(t), \zeta_n(t)) \to (-\dot{u}(t) + z(t), z(t))$ strongly in H, and on the other hand the inclusions (2.12) and (2.14) hold true for every integer n as well as the inclusions

$$
-\dot{u}(t) + z(t) \in \bigcap_n \overline{\mathrm{co}}\Big\{ - \dot{u}_q(t) + z_q(t) : q \ge n\Big\} \text{ and } z(t) \in \bigcap_n \overline{\mathrm{co}}\Big\{z_q(t) : q \ge n\Big\}.
$$

It results from (2.12) and (2.14) that for any $n \in \mathbb{N}$, any $t \in [0,T] \setminus S$, and for any $y \in H$

$$
\big\langle y, -\dot{u}_n(t) + z_n(t)\big\rangle \le \sigma\bigg(y, \gamma\partial d_{C\left(\theta_n(t), u_n\left(\delta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big)\bigg) \tag{2.16}
$$

and

$$
\big\langle y, z_n(t)\big\rangle \le \sigma\bigg(y, G\Big(\delta_n(t), u_n\big(\delta_n(t)\big)\Big)\bigg). \tag{2.17}
$$

Further, for each $n \in \mathbb{N}$ and any $t \in [0,T] \backslash S$, from (2.15) we have

$$\big\langle y, \xi_k(t)\big\rangle \le \sup_{q\ge n} \big\langle y, -\dot{u}_q(t) + z_q(t)\big\rangle \text{ for all } k \ge n$$

and

$$\big\langle y, \zeta_k(t)\big\rangle \le \sup_{q\ge n} \big\langle y, z_q(t)\big\rangle \text{ for all } k \ge n,$$

and taking the limit in both inequalities as $k \to +\infty$ gives through (2.16) and (2.17)

$$\begin{aligned}\big\langle y, -\dot{u}(t) + z(t)\big\rangle &\le \sup_{q\ge n} \big\langle y, -\dot{u}_q(t) + z_q(t)\big\rangle \\ &\le \sup_{q\ge n} \sigma\Big(y, \gamma\partial d_{C\big(\theta_q(t), u_q\big(\delta_q(t)\big)\big)}\big(u_q\big(\theta_q(t)\big)\big)\Big)\end{aligned}$$

and

$$\big\langle y, z(t)\big\rangle \le \sup_{q\ge n} \big\langle y, z_q(t)\big\rangle \le \sup_{q\ge n} \sigma\Big(y, G\Big(\delta_q(t), u_q\big(\delta_q(t)\big)\Big)\Big),$$

which ensures that

$$\big\langle y, -\dot{u}(t) + z(t)\big\rangle \le \limsup_{n\to+\infty} \sigma\Big(y, \gamma\partial d_{C\big(\theta_n(t), u_n\big(\delta_n(t)\big)\big)}\big(u_n\big(\theta_n(t)\big)\big)\Big)$$

and

$$\big\langle y, z(t)\big\rangle \le \limsup_{n\to+\infty} \sigma\Big(y, G\Big(\delta_n(t), u_n\big(\delta_n(t)\big)\Big)\Big).$$

Observe also by $(\mathcal{H}_3)$ and Proposition 1.4 that the multimapping

$$\operatorname{gph} C \ni (t, x, x') \mapsto \partial d_{C(t,x)}(x')$$

takes on weakly compact convex values and is upper semicontinuous from $\operatorname{gph} C$ into $(H, w(H,H))$, hence for each $y \in H$ the restriction to $\operatorname{gph} C$ of the real-valued function

$$(t, x, x') \mapsto \sigma\big(y, \gamma\partial d_{C(t,x)}(x')\big)$$

is upper semicontinuous on $\operatorname{gph} C$. Further, the extended real-valued function $(t,x) \mapsto \sigma(y, G(t,x))$ is also upper semicontinuous on $[0,T] \times H$ by assumption $(\mathcal{H}_1)$. Since

$$\big(\theta_n(t), u_n(\delta_n(t)), u_n(\theta_n(t)\big) \in \operatorname{gph} C \quad \text{for all } n,$$

it follows from the two latter inequalities that, for every $t \in [0,T] \setminus S$ and every $y \in H$,

$$\big\langle y, -\dot{u}(t) + z(t)\big\rangle \le \sigma\Big(y, \gamma\partial d_{C\big(t, u(t)\big)}\big(u(t)\big)\Big)$$

and

$$\big\langle y, z(t)\big\rangle \le \sigma\Big(y, G\big(t, u(t)\big)\Big),$$

which ensures that $-\dot{u}(t)+z(t) \in \gamma\partial d_{C(t,u(t))}(u(t))$ and $z(t) \in G(t,u(t))$, and consequently

$$\dot{u}(t) \in -N_{C\big(t,u(t)\big)}\big(u(t)\big)+z(t) \text{ a.e. },$$

$$z(t) \in G\big(t,u(t)\big) \text{ a.e. },$$

with

$$\|\dot{u}(t)-z(t)\| \le \gamma.$$

The proof is then complete. $\square$

We turn now to the case where (in place of $(\mathcal{H}_3)$) the variation of C with respect to the time t is absolutely continuous, that is, there exists an absolutely function ϑ on $[0,T]$ with $\vartheta(0)=0$ such that

$$(\mathcal{H}_3') \qquad |d\big(y,C(t,x)\big)-d\big(y',C(s,x')\big)| \le \|y-y'\|+|\vartheta(t)-\vartheta(s)|+L\|x-x'\|$$

for all $t,s \in [0,T]$, x,x',y,y' in H. Note that, if there is an absolutely continuous function $v:[0,T]\to\mathbb{R}$ satisfying the above inequality, putting $\vartheta(t)=\int_0^t\big(|\dot{v}(\tau)|+\varepsilon\big)\,d\tau$, the function ϑ fulfills the same inequality as well as the conditions $\vartheta(0)=0$ and $\dot{\vartheta}(\cdot)\ge\omega$ with $\omega:=\varepsilon>0$.

Theorem 2.2 *Assume that H is a Hilbert space, that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3')$, and $(\mathcal{H}_4)$ hold. Then, for any $u_0 \in H$ with $u_0 \in C(0,u_0)$, there exists an absolutely continuous solution $u:[0,T]\to H$ of the differential inclusion $(\mathcal{D})$.*

Proof. According to what precedes the statement of the theorem, we may and do suppose (in addition to the equality $\vartheta(0)=0$) that there is some real $\omega>0$ such that $\dot{\vartheta}(\cdot)\ge\omega>0$. The absolutely continuous function ϑ is then increasing and admits an increasing continuous inverse $\vartheta^{-1}:[0,\widehat{T}]\to[0,T]$, where $\widehat{T}:=\vartheta(T)$. Further, ϑ^{-1} is Lipschitz. Indeed, taking $\hat{s}=\vartheta(s)$, $\hat{t}=\vartheta(t)$ in $[0,\widehat{T}]$ with $\hat{s}\le\hat{t}$ we can write

$$\vartheta^{-1}(\hat{t})-\vartheta^{-1}(\hat{s})=t-s\le\omega^{-1}\int_s^t\dot{\vartheta}(r)\,dr=\omega^{-1}\big(\vartheta(t)-\vartheta(s)\big)=\omega^{-1}|\hat{t}-\hat{s}|.$$

Now for each $\tau\in[0,\widehat{T}]$ put

$$\widehat{C}(\tau,x):=C\big(\vartheta^{-1}(\tau),x\big) \quad \text{and} \quad \widehat{G}(\tau,x):=G\big(\vartheta^{-1}(\tau),x\big)$$

and note that

$$\begin{aligned}|d\big(y,\widehat{C}(\tau,x)\big)-d\big(y,\widehat{C}(\tau',x')\big)| &\le |\vartheta^{-1}(\tau)-\vartheta^{-1}(\tau')|+L_2\|x-x'\|\\ &\le \omega^{-1}|\tau-\tau'|+L_2\|x-x'\|.\end{aligned}$$

By Theorem 2.1 the differential inclusion $(\widehat{\mathcal{D}})$ associated with $\widehat{C}$ and $\widehat{G}$ in place of C and G and with initial condition $u_0\in\widehat{C}(0,u_0)$ (note that $\widehat{C}(0,u_0)=C(0,u_0)$) admits at least a Lipschitz solution. Let U be a solution of $(\widehat{\mathcal{D}})$ on $[0,\widehat{T}]$ and set

$u(t) := U(\vartheta(t))$ for all $t \in [0,T]$. This mapping u is clearly absolutely continuous and for almost all $t \in [0,T]$ we easily see (in a standard way) that

$$\frac{du}{dt}(t) = \frac{d\vartheta}{dt}(t)\,\dot{U}\big(\vartheta(t)\big). \tag{2.18}$$

From this and what precedes it is not difficult (as, e.g., in the proof of [13, Theorem 5.1]) to derive that u is a solution of $(\mathcal{D})$. $\square$

Assuming as in the above proof that $\dot{\vartheta}(\cdot) \geq \omega > 0$, then (2.18) combined with Theorem 2.1 easily provides estimate for $\big\|\frac{du}{dt}(t)\big\|$.

The next theorem proves the existence of solution on the whole interval $\mathbb{R}_+ := [0,+\infty[$. This requires to replace the above assumptions $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3', \mathcal{H}_4$ by $\mathcal{G}_1, \cdots, \mathcal{G}_4$ when the time describes $\mathbb{R}_+$. In such a context the solution will be locally absolutely continuous.

Theorem 2.3 *Let $G : \mathbb{R}_+ \times H \rightrightarrows H$ be a multimapping which is scalarly upper semicontinuous with respect to both variables. Assume that H is a Hilbert space, that $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ below hold:*

$(\mathcal{G}_1)$ *There exists a non-negative function $\beta(\cdot) \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\mathbb{R}_+)$ such that*

$$d\big(0, G(t,x)\big) \leq \beta(t)$$

for all $t \in \mathbb{R}_+$ and $x \in H$;

$(\mathcal{G}_2)$ *For all $t \in \mathbb{R}_+$ and $x \in H$, the sets $C(t,x)$ are nonempty closed in H and equi-uniformly subsmooth (with parameter $(t,x) \in \mathbb{R}_+ \times H$);*

$(\mathcal{G}_3)$ *There are a real constant $L_2 \in [0,1[$ and a locally absolutely continuous function ϑ on $\mathbb{R}_+$ such that, for all $t,s \in \mathbb{R}_+$ and $x,x',y,y' \in H$*

$$\big|d\big(y, C(t,x)\big) - d\big(y', C(s,x')\big)\big| \leq \|x-y\| + |\vartheta(t) - \vartheta(s)| + L_2\|x-x'\|;$$

$(\mathcal{G}_4)$ *For any real $\tau > 0$ and any bounded subset $A \subset H$, the set $C([0,\tau] \times A)$ is relatively ball-compact.*

Then, given $u_0 \in H$ with $u_0 \in C(0,u_0)$, there exists a mapping $u : \mathbb{R}_+ \to H$ which is locally absolutely continuous on $\mathbb{R}_+$ and satisfies

$$(\mathcal{D}_{\mathbb{R}_+}) \quad \begin{cases} \dot{u}(t) \in -N_{C\left(t,u(t)\right)}\big(u(t)\big) + G\big(t,u(t)\big) \quad \text{a.e. } t \in \mathbb{R}_+, \\[2ex] u(t) \in C\big(t,u(t)\big) \ \ \forall t \in \mathbb{R}_+, \\[2ex] u(t) = u_0 + \int_0^t \dot{u}(s)ds \ \ \forall t \in \mathbb{R}_+. \end{cases}$$

Proof. Put $T_k = k$ for all $k \in \{0\} \cup \mathbb{N}$. It will suffice to prove that Theorem 2.2 applies on each interval $[T_k, T_{k+1}]$.

According to assumptions $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ we have that $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}'_3, \mathcal{H}_4$ hold on the interval $[T_0, T_1]$. Since $u_0 \in C(T_0, u_0)$, by Theorem 2.3 there exists an absolutely continuous mapping $u^0 : [T_0, T_1] \to H$ such that

$$\begin{cases} \dot{u}^0(t) \in -N_{C\left(t,u^0(t)\right)}\big(u^0(t)\big) + G\big(t, u^0(t)\big) \quad \text{a.e } \ t \in [T_0, T_1], \\ \\ u^0(t) \in C\big(t, u^0(t)\big) \ \forall t \in [T_0, T_1], \\ \\ u^0(T_0) = u_0. \end{cases}$$

Suppose $u^0, \cdots, u^{k-1}$ have been constructed such that, for $l = 0, \cdots, k-1$, $u^l : [T_l, T_{l+1}] \to H$ is absolutely continuous, $u^l(T_l) = u^{l-1}(T_l)$, $u^l(t) \in C(t, u^l(t))$ for all $t \in [T_l, T_{l+1}]$ and

$$\dot{u}^l(t) \in -N_{C\left(t,u^l(t)\right)}\big(u^l(t)\big) + G\big(t, u^l(t)\big) \quad \text{a.e } \ t \in [T_l, T_{l+1}].$$

In an analogous way as above, the hypotheses $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ ensure that $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}'_3, \mathcal{H}_4$ hold on the interval $[T_k, T_{k+1}]$ and we have $u^{k-1}(T_k) \in C(T_k, u^{k-1}(T_k))$. It follows from Theorem 2.2 that there is an absolutely continuous mapping $u^k : [T_k, T_{k+1}] \to H$ such that

$$\begin{cases} \dot{u}^k(t) \in -N_{C\left(t,u^k(t)\right)}\big(u^k(t)\big) + G\big(t, u^k(t)\big) \quad \text{a.e } \ t \in [T_k, T_{k+1}], \\ \\ u^k(t) \in C\big(t, u^k(t)\big) \ \forall t \in [T_k, T_{k+1}], \\ \\ u^k(T_k) = u^{k-1}(T_k). \end{cases} \tag{2.19}$$

So, we obtain by induction u^k for all $k \in \{0\} \cup \mathbb{N}$ with the above properties.

Let $u : \mathbb{R}_+ \to H$ be the mapping defined by

$$u(t) := u^k(t) \quad \text{ for all } t \in [T_k, T_{k+1}[\text{ with } k \in \{0\} \cup \mathbb{N}.$$

It is easily seen that u is locally absolutely continuous on $\mathbb{R}_+$. Therefore, it results from (2.19) that

$$\begin{cases} \dot{u}(t) \in -N_{C\left(t,u(t)\right)}\big(u(t)\big) + G\big(t, u(t)\big) \quad \text{a.e } \ t \in \mathbb{R}_+, \\ \\ u(t) \in C\big(t, u(t)\big) \ \forall t \in \mathbb{R}_+, \\ \\ u(0) = u^0(T_0) = u_0. \end{cases}$$

This finishes the proof of the theorem. $\square$

As direct consequences of Theorem 2.2 and Theorem 2.3 we obtain below Corollary 2.1 and Corollary 2.2 respectively.

Corollary 2.1 *Let $G : [0,T] \times H \rightrightarrows H$ be a multimapping which is scalarly upper semicontinuous with respect to both variables. Assume that H is a finite dimensional Euclidean space and that the assumptions below hold:*

- *There exists a positive real number α such that*
$$d\big(0, G(t,x)\big) \leq \alpha$$
for all $t \in [0,T]$ and $x \in H$;
- *For all $t \in [0,T]$ and each $x \in H$, the sets $C(t,x)$ are nonempty closed in H and equi-uniformly subsmooth (with $(t,x) \in [0,T] \times H$ as parameter);*
- *There are a real constant $L_2 \in [0,1[$ and an absolutely continuous function ϑ on $[0,T]$ such that, for all $t, s \in [0,T]$ and $x, x', y, y' \in H$*
$$\big|d\big(y, C(t,x)\big) - d\big(y', C(s,x')\big)\big| \leq \|y - y'\| + |\vartheta(t) - \vartheta(s)| + L_2\|x - x'\|.$$

Then, given $u_0 \in H$ with $u_0 \in C(0,u_0)$, there exists a mapping $u : [0,T] \to H$ which is absolutely continuous on $[0,T]$ and satisfies $(\mathcal{D})$.

Corollary 2.2 *Let $G : \mathbb{R}_+ \times H \rightrightarrows H$ be a multimapping which is scalarly upper semicontinuous with respect to both variables. Assume that H is a finite dimensional Euclidean space and that the following assumptions hold:*

- *There exists a non-negative function $\beta(\cdot) \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\mathbb{R}_+)$ such that*
$$d\big(0, G(t,x)\big) \leq \beta(t)$$
for all $t \in \mathbb{R}_+$ and $x \in H$;
- *For all $t \in \mathbb{R}_+$ and each $x \in H$, the sets $C(t,x)$ are nonempty closed in H and equi-uniformly subsmooth (with $(t,x) \in \mathbb{R}_+$ as paremeter);*
- *There are a real constant $L_2 \in [0,1[$ and a locally absolutely continuous function ϑ on $\mathbb{R}_+$ such that, for all $t, s \in \mathbb{R}_+$ and $x, x', y, y' \in H$*
$$\big|d\big(y, C(t,x)\big) - d\big(y', C(s,x')\big)\big| \leq \|y - y'\| + |\vartheta(t) - \vartheta(s)| + L_2\|x - x'\|.$$

Then, given $u_0 \in H$ with $u_0 \in C(0,u_0)$, there exists a mapping $u : \mathbb{R}_+ \to H$ which is locally absolutely continuous on $\mathbb{R}_+$ and satisfies $(\mathcal{D}_{\mathbb{R}_+})$.

Finally, when H is separable, in all the above results, the scalar upper semicontinuity of G with respect to both variables can be replaced, through appropriate reformulations, by the separate measurability of G with respect to t and the scalar upper semicontinuity with respect to x. Indeed, using a growth condition in place of $(\mathcal{H}_1)$ it suffices as in [5] (see also [13, Theorem 4.1]) to apply the version of Scorza-Dragoni theorem (for separately measurable and scalarly upper semicontinuous multimapping) and the version of Dugundji extension theorem (for upper semicontinuous multimapping) (see, e.g., [3]).

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