For a real function $f:\mathbb{R}\to\mathbb{R}$ we denote by $f'_+(t)$ the right derivative of f at t when it exists, that is,

$$f'_+(t)=\lim_{r\downarrow 0}\frac{f(t+r)-f(t)}{r}.$$

Lemma 0.1 *Let $h,v\in X$, with $\|v\|>1$ and $\|h\|=1$ and v and h are linearly independent, $\beta>1$ and $s\geq 0$. Consider the functions $f,g:\mathbb{R}\to\mathbb{R}$ defined by*

$$f(t)=\|tv+\beta h\|-\|tv+2sv+h\|,\quad g(t)=\|tv-\beta h\|-\|tv+2sv-h\|.$$

Then f and g are differentiable and satisfying :

- *f is non-decreasing on $]-\infty,-\frac{2s\beta}{\beta-1}]$ ($(f'_+t)\geq 0$, $\forall t\leq 0$) and non-increasing on $[-\frac{2s\beta}{\beta-1},+\infty[$ ($(f'_+t)\leq 0$, $\forall t\geq -\frac{2s\beta}{\beta-1}$).*
- *g is non-decreasing on $]-\infty,-\frac{2s\beta}{\beta-1}]$ ($(g't)\geq 0$, $\forall t\leq -\frac{2s\beta}{\beta-1}$) and non-increasing on $[-\frac{2s\beta}{\beta-1},+\infty[$ ($(g't)\leq 0$, $\forall t\geq -\frac{2s\beta}{\beta-1}$).*
- *for all $t\geq 0$*

$$f(t)\geq(\beta-1)\nu'(v;h)-2s\|v\|,\quad \text{and}\quad g(t)\geq(\beta-1)\nu'(v;-h)-2s\|v\|,$$

where $\nu'(v;\cdot)$ denotes the usual directional derivative of the convex function $\nu:=\|\ \|$.

Proof. We study only the function f since the result for g is obtained in changing h in $-h$. It is not difficult to see that $\nu'(ru;v)=\nu'(u;v)$ for all $u,v\in X$. So, for all $t\neq 0$,

$$f'_+(t)=\nu'(tv+\beta h;v)-\nu'(tv+2sv+h;v)=\nu'(\frac{t}{\beta}v+h;v)-\nu'(tv+2sv+h;v).$$

From the continuity of the convex function ν, choose $x_1^*\in\partial\nu(\frac{t}{\beta}v+h)$ and $x_2^*\in\partial\nu(tv+2sv+h)$ such that

$$\langle x_1^*,v\rangle=\nu'(\frac{t}{\beta}v+h;v)\quad\text{and}\quad\langle x_2^*,v\rangle=\nu'(tv+2sv+h;v).$$

Consequently,

$$f'_+(t)=\frac{\beta}{(1-\beta)t-2s\beta}\langle x_1^*-x_2^*,(\frac{t}{\beta}v+h)-(tv+2sv+h)\rangle,$$

hence the monotonicity of the subdifferential $\partial\nu$ ensures that

$$f'_+(t)\geq 0\quad\text{for all } t\leq-\frac{2s\beta}{\beta-1}\quad\text{and}\quad f'_+(t)\leq 0\quad\text{for all } t\geq-\frac{2s\beta}{\beta-1}.$$

We deduce that f is non-decreasing on $]-\infty,-\frac{2s\beta}{\beta-1}]$ and non-increasing on $[-\frac{2s\beta}{\beta-1},+\infty[$. The last item results from the following

$$\begin{aligned}\lim_{t\to+\infty}f(t)&\geq-2s\|v\|+\lim_{\tau\to 0^+}\frac{\|v+\tau\beta h\|-\|v\|}{\tau}-\frac{\|v+\tau h\|-\|v\|}{\tau}\\&=-2s\|v\|+\beta\nu'(v;h)-\nu'(v;h)=(\beta-1)\nu'(v;h)-2s\|v\|.\end{aligned}$$

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The following lemma whose proof is similar to that of Lemma 2.2 in [1] will be used later.

Lemma 0.2 *Let $\varepsilon>0$, $u,h\in X$, with $\|u\|=1$ and $0<\|h\|\leq\min(1,\frac{\varepsilon}{4})$, satisfying*

$$\|u+h\|>\|u-h\|,\quad\frac{\|u+h\|+\|u-h\|-2}{\|h\|}\geq\varepsilon.\tag{0.1}$$

Then there exists $s\in]0,\|h\|]$ such that

- *$\|u+h-su\|=\|u-h+su\|$ and*
- *$\frac{\|u+h-su\|-1}{\|h-su\|}\geq\frac{\varepsilon}{8}$.*

Theorem 0.1 *Let X be a Banach space. Then the following assertions are equivalent :*

1. *The norm $\|\cdot\|_*$ of X^* is W*UR.*
2. *For each closed set $S\subset X$, and all $a,b\in S$, with $a\neq b$, we have*
$$\liminf_{\|x\|\to+\infty}\left(d_{[a,b]}(x)-d_S(x)\right)\geq 0. \tag{0.2}$$
3. *For all $a,b\in X$, with $a\neq b$, we have*
$$\liminf_{\|x\|\to+\infty}\left(d_{[a,b]}(x)-\min\{\|x-a\|,\|x-b\|\}\right)\geq 0. \tag{0.3}$$
4. *The norm $\|\cdot\|$ is UGD.*

Comment. We know that 1. $\Longleftrightarrow$ 4. (see the book by Deville, Godefroy and Zisler) and 4. $\Longrightarrow$ 2. (this is exactly our Proposition 4.2). Here we give another proof of the latter implication.

Proof. 1. $\Longrightarrow$ 2. : Suppose that the assertion 2. is not satisfied. Then there exist $\mu>0$, a closed set $S\subset X$, and $a,b\in S$, with $a\neq b$, and a sequence $(x_n)\subset X$, with $\|x_n\|\to+\infty$ such that for n large enough
$$\lim_{\|n\|\to+\infty}\left(d_{[a,b]}(x_n)-d_S(x_n)\right)<-\mu. \tag{0.4}$$
Fix n sufficiently large satisfying (0.4). Let $p_n\in[a,b]$ such that $d_{[a,b]}(x_n)=\|x_n-p_n\|$. Fix $x_n^*\in-\partial d_{[a,b]}(p_n)$, so $\|x_n^*\|=1$ for n large enough since $x_n\notin[a,b]$ according to $\|x_n\|\to+\infty$. Observe that, for all $p\in[a,b]$,
$$\langle -x_n^*,p-x_n\rangle\leq d_{[a,b]}(p)-d_{[a,b]}(x_n)=-d_{[a,b]}(x_n). \tag{0.5}$$
For all $p\in[a,b]$, it ensues that
$$\begin{aligned}\langle -x_n^*,p-p_n\rangle &= \langle -x_n^*,p-x_n\rangle+\langle -x_n^*,x_n-p_n\rangle\\ &\quad -d_{[a,b]}(x_n)+\|x_n-p_n\|\\ &= -d_{[a,b]}(x_n)+d_{[a,b]}(x_n)=0.\end{aligned}$$
This and the inclusion $p_n\in]a,b[$ for n large enough (due (0.4)) $p_n\in]a,b[$ entail
$$\langle x_n^*,a-b\rangle=0. \tag{0.6}$$
Further, from (0.5) we also have
$$\langle -x_n^*,p_n-x_n\rangle\leq -d_{[a,b]}(x_n)=-\|x_n-p_n\|$$
hence
$$\|x_n-p_n\|\leq\langle x_n^*,p_n-x_n\rangle. \tag{0.7}$$
On the onther hand, since $\|p_n-x_n\|-\|a-x_n\|\leq-\mu,$ then, for all $a_n^*\in\partial\|\cdot-x_n\|(a)$, we have
$$\mu\leq\|a-x_n\|-\|p_n-x_n\|\leq\langle a_n^*,a-p_n\rangle. \tag{0.8}$$
Let $t_n\in]0,1[$ be such that $p_n=t_na+(1-t_n)b$. Using relation (0.4), we find $s_1,s_2\in]0,1[$ such that for all n sufficiently large $s_1<t_n<s_2$. As $a-p_n=(1-t_n)(a-b)$, relation (0.8) ensures that
$$\frac{\mu}{1-s_1}\leq\frac{\mu}{1-t_n}\leq\langle a_n^*,a-b\rangle$$
which implies (via (0.6))
$$\frac{\mu}{1-s_1}\leq\langle a_n^*-x_n^*,a-b\rangle. \tag{0.9}$$
On the other hand, relations (0.7) and (0.6) ensure that
$$\begin{aligned}\|p_n-x_n\|+\|a-x_n\| &\leq \langle x_n^*,p_n-x_n\rangle+\langle a_n^*,a-x_n\rangle\\ &= \langle x_n^*,t_n(a-b)+b-x_n\rangle+\langle a_n^*,a-x_n\rangle\\ &= \langle x_n^*,b-x_n\rangle+\langle a_n^*,a-x_n\rangle\\ &= \langle x_n^*,a-x_n\rangle+\langle a_n^*,a-x_n\rangle\\ &= \langle x_n^*+a_n^*,a-x_n\rangle\\ &\leq \|x_n^*+a_n^*\|_*\cdot\|a-x_n\|.\end{aligned}$$

Hence

$$\frac{\|p_n - x_n\| + \|a - x_n\|}{\|a - x_n\|} \le \|x_n^* + a_n^*\|.$$

Thus

$$\|x_n^*\| = \|a_n^*\| = 1, \quad \lim_{n\to+\infty} \|x_n^* + a_n^*\|_* = 2 \tag{0.10}$$

and by (0.9) we have

$$\liminf_{n\to+\infty} \langle a_n^* - x_n^*, a - b\rangle \ge \frac{\mu}{1 - s_1}. \tag{0.11}$$

It results that relations (0.10) and (0.11) contradict the W*UR property of X^*.

2. $\Longrightarrow$ 3. : It is obvious.

3. $\Longrightarrow$ 4. : **NOTE THAT relation 3. ensures the Gâteaux differentiability of the norm. OK**

Suppose that the norm $\|\cdot\|$ is not UGD. Then there exist $h \in X$, with $\|h\| = 1$, $\varepsilon > 0$ and sequences $(u_n) \subset X$ with $\|u_n\| = 1$ for all n and $(t_n) \subset]0, +\infty[$ with $\lim_{n\to\infty} t_n = 0$ such that, for all $n \in \mathbb{N}$,

$$\frac{\|u_n + t_n h\| + \|u - t_n h\| - 2}{t_n} \ge \varepsilon \tag{0.12}$$

The proof of this implication is down in two steps.

Step1. For infinitely many integers n, the equality

$$\|u_n + t_n h\| = \|u_n - t_n h\|$$

holds true. Put $a = h$ and $b = -h$ and $x_n = \frac{u_n}{t_n}$. Then

$$\lim_{n\to+\infty} \|x_n\| = +\infty,\ d_{[a,b]}(x_n) \le \|x_n\|$$

and

$$\min\{\|x_n - a\|, \|x_n - b\|\} = \|x_n - a\|.$$

Relation (0.12) ensures that

$$d_{[a,b]}(x_n) - \min\{\|x_n - a\|, \|x_n - b\|\} \le \|x_n\| - \|x_n - a\| = \frac{1}{t_n} - \frac{1}{t_n}\|u_n - t_n h\|$$
$$= \frac{1}{2}\{\frac{2 - \|u_n - t_n h\| - \|u_n + t_n h\|}{t_n}\} \le -\frac{\varepsilon}{2}$$

and this contradicts the assumption 3.

Step2. The equality holds true only for a finite number of integers n. Without loss of generality, suppose that

$$\forall n \in \mathbb{N}, \quad \|u_n + t_n h\| > \|u_n - t_n h\|.$$

Applying Lemma 0.2, with $h = t_n h$ and $u = u_n$, there exists $s_n \in]0, t_n[$ such that

$$\|u_n + t_n h - s_n u_n\| = \|u_n - t_n h + s_n u_n\| \tag{0.13}$$

$$\frac{\|u_n + t_n h - s_n u_n\| - 1}{\|t_n h - s_n u_n\|} \ge \frac{\varepsilon}{8}. \tag{0.14}$$

Relations (0.13) and (0.14) assert that u_n and h are linearly dependent for at most a finite number of integers n (THIS NEEDS TO BE ARGUED). So we may assume that for all $n \in \mathbb{N}$, u_n and h are linearly independent.

Put $v_n = \frac{u_n}{t_n}$ and $x_n = (1 - s_n)v_n$. Two cases may occur:

Case I : $\liminf_{n\to+\infty} \|h - s_n v_n\| = 0$. Without loss of generality we may assume that $\lim_{n\to+\infty} \|h - s_n v_n\| = 0$. In this case, since $\|h\| = \|u_n\| = 1$,

$$\lim_{n\to+\infty} \frac{s_n}{t_n} = 1. \tag{0.15}$$

Put $a = \frac{3}{2}h$ and $b = -\frac{3}{2}h$. Apply Lemma 0.1, with $\beta := \frac{3}{2}$, $t := 1 - s_n$ and $v = v_n$, the last item of this lemma gives, with $s := 0$ in the first inequality of that item and $s := s_n$ in the second inequality,

$$\|(1-s_n)v_n + \frac{3}{2}h\| \geq \|(1-s_n)v_n + h\| + \frac{1}{2}\nu'(v_n;h)$$

and

$$\|(1-s_n)v_n - \frac{3}{2}h\| \geq \|(1+s_n)v_n - h\| + \frac{-1}{2}\nu'(v_n;h) - 2s_n\|v_n\|.$$

Taking into account relation (0.13), we obtain

$$\|x_n - a\| \geq \|(1+s_n)v_n - h\| + \frac{-1}{2}\nu'(v_n;h) - 2s_n\|v_n\| \geq \|(1+s_n)v_n - h\| - \frac{1}{2} - 2s_n\|v_n\| \tag{0.16}$$

and

$$\|x_n - b\| \geq \|(1+s_n)v_n - h\| + \frac{1}{2}\nu'(v_n;h) \geq \|(1+s_n)v_n - h\| - \frac{1}{2}. \tag{0.17}$$

Consequently

$$\min\{\|x_n - a\|, \|x_n - b\|\} \geq \|(1+s_n)v_n - h\| - \frac{1}{2} - 2s_n\|v_n\|.$$

Since $d_{[a,b]}(x_n) \leq \|x_n\| = \frac{1-s_n}{t_n}$, relations (0.14) and (0.16) ensure that

$$\begin{aligned} d_{[a,b]}(x_n) - \min\{\|x_n - a\|, \|x_n - b\|\} &\leq \frac{1-s_n}{t_n} - \|(1+s_n)v_n - h\| + \frac{1}{2} + 2s_n\|v_n\| \\ &\leq -\frac{\varepsilon}{8}\|h - \frac{s_n}{t_n}u_n\| - \frac{s_n}{t_n} + \frac{1}{2} + 2s_n\|v_n\| \\ &= -\frac{\varepsilon}{8}\|h - \frac{s_n}{t_n}u_n\| + \frac{s_n}{t_n} + \frac{1}{2}. \end{aligned}$$

FROM THE LATTER NO DIRECT CONTRADICTION IS OBTAINED

Case II: $\liminf_{n\to+\infty} \|h - s_n v_n\| = \alpha > 0$. Without loss of generality we may assume that $\lim_{n\to+\infty} \|h - s_n v_n\| = \alpha$.

Put $\beta = 1 + \frac{\varepsilon\alpha}{16}$, $a = \beta h$ and $b = -\beta h$. Apply once again Lemma 0.1, with $s = s_n$ and $v = v_n$, and taking into account relation (0.13) we get from the last item of this lemma with $t := 1 - s_n$ and $s := s_n$,

$$\|x_n - a\| \geq \|(1+s_n)v_n - h\| + (1-\beta)\nu'(v_n;h) - 2s_n\|v_n\| \tag{0.18}$$

and

$$\|x_n - b\| \geq \|(1+s_n)v_n - h\| + (\beta-1)\nu'(v_n;h). \tag{0.19}$$

It ensues that

$$\min\{\|x_n - a\|, \|x_n - b\|\} \geq \|(1+s_n)v_n - h\|(1-\beta) - 2s_n\|v_n\|.$$

Since $d_{[a,b]}(x_n) \leq \|x_n\| = \frac{1-s_n}{t_n}$, relations (0.14) and (0.18) ensure that

$$\begin{aligned} d_{[a,b]}(x_n) - \min\{\|x_n - a\|, \|x_n - b\|\} &\leq \frac{1-s_n}{t_n} - -\|(1+s_n)v_n - h\| + (\beta-1) + 2s_n\|v_n\| \\ &= -\frac{\varepsilon}{8}\|h - \frac{s_n}{t_n}u_n\| - \frac{s_n}{t_n} + (\beta-1) + 2s_n\|v_n\| \\ &\leq -\frac{\varepsilon}{8}\|h - \frac{s_n}{t_n}u_n\| + \frac{s_n}{t_n} + \frac{\varepsilon\alpha}{16} \end{aligned}$$

HERE ALSO NO CONTRADICTION IS DIRECTLY OBTAINED

References

[1] D. J. Ives, BUMP FUNCTIONS AND DIFFERENTIABILITY IN BANACH SPACES, Proceeding AMS, 129 (2001), 3583-3588.